

## GAPS BETWEEN PRIMES IN BEATTY SEQUENCES

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ABSTRACT. In this paper, we study the gaps between primes in Beatty sequences following the methods in the recent breakthrough of [9].

2010 Mathematics Subject Classification: 11B05, 11L20, 11N35

## 1. INTRODUCTION

Let  $p_n$  denote the  $n$ -th prime and  $t$  a natural number with  $t \geq 2$ . It has long been conjectured that

$$\liminf_{n \rightarrow \infty} (p_{n+t-1} - p_n) < \infty.$$

This was established recently for  $t = 2$  by Y. Zhang [12] and shortly after for all  $t$  by J. Maynard [9]. Maynard showed that for  $N > C(t)$ , the interval  $[N, 2N)$  contains a set  $\mathcal{S}$  of  $t$  primes of diameter

$$D(\mathcal{S}) \ll t^3 \exp(4t),$$

where

$$D(\mathcal{S}) := \max\{n : n \in \mathcal{S}\} - \min\{n : n \in \mathcal{S}\}.$$

In the present paper, we adapt Maynard's method to prove a similar result where  $\mathcal{S}$  is contained in a prescribed set  $\mathcal{A}$  (see Theorem 1). We then work out applications (Theorems 2 and 3) to a section of a Beatty sequence, so that

$$\mathcal{A} = \{[\alpha m + \beta] : m \geq 1\} \cap [N, 2N).$$

The number  $\alpha$  is assumed to be irrational with  $\alpha > 1$ , while  $\beta$  is a given real number. We require an auxiliary result (Theorem 4) for the estimation of errors of the form

$$\sum_{\substack{N \leq n < N' \\ \gamma n \in I \pmod{1} \\ n \equiv a \pmod{q}}} \Lambda(n) - \frac{(N - N')|I|}{\varphi(q)},$$

where  $I$  is an interval of length  $|I| < 1$  and  $\gamma = \alpha^{-1}$ . Theorem 4 is of “Bombieri-Vinogradov type”; for completeness, we include a result of Barban-Davenport-Halberstam type for these errors (Theorem 5).

We note that Chua, Park and Smith [5] have already used Maynard's method to prove the existence of infinitely many sets of  $k$  primes of diameter at most  $C = C(\alpha, k)$  in a Beatty sequence  $[\alpha n]$ , where  $\alpha$  is irrational and of finite type. However, no explicit bound for  $C$  is given.

In this paragraph, we introduce some notations to be used throughout this paper. We suppose that  $t \in \mathbb{N}$ ,  $N \geq C(t)$  and write  $\mathcal{L} = \log N$ ,

$$D_0 = \frac{\log \mathcal{L}}{\log \log \mathcal{L}}.$$

Moreover,  $(d, e)$  and  $[d, e]$  stand for the great common divisor and the least common multiple of  $d$  and  $e$ , respectively.  $\tau(q)$  and  $\tau_k(q)$  are the usual divisor functions.  $\|x\|$  is the distance of between  $x \in \mathbb{R}$  and the

nearest integer. Set

$$P(z) = \prod_{p < z} p \text{ with } z \geq 2 \text{ and } \psi(n, z) = \begin{cases} 1 & \text{if } (n, P(z)) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

$X(E; n)$  stands for the indicator function of a set  $E$  and  $\mathbb{P}$  for the set of primes. Let  $\varepsilon$  be a positive constant, sufficiently small in terms of  $t$ . The implied constant “ $\ll$ ”, when it appears, may depend on  $\varepsilon$  and on  $A$  (if  $A$  appears in the statement of the result). “ $F \asymp G$ ” means both  $F \ll G$  and  $G \ll F$  hold. As usual,  $e(y) = \exp(2\pi i y)$ , and  $o(1)$  indicates a quantity tending to 0 as  $N$  tends to infinity. Furthermore,

$$\sum_{\chi \bmod q}, \quad \sum'_{\chi \bmod q}, \quad \sum^*_{\chi \bmod q}$$

denote, respectively, a sums over all Dirichlet characters modulo  $q$ , a sum over nonprincipal characters modulo  $q$  and a sum restricted to primitive characters, other than  $\chi = 1$ , modulo  $q$ . We write  $\hat{\chi}$  for the primitive character that induces  $\chi$ . A set  $\mathcal{H} = \{h_1, \dots, h_k\}$  of distinct non-negative integers is *admissible* if for every prime  $p$ , there is an integer  $a_p$  such that  $a_p \not\equiv h \pmod{p}$  for all  $h \in \mathcal{H}$ .

In Sections 1 and 2, let  $\theta$  be a positive constant. Let  $\mathcal{A}$  be a subset of  $[N, 2N] \cap \mathbb{N}$ . Suppose that  $Y > 0$  and  $Y/q_0$  is an approximation to the cardinality of  $\mathcal{A}$ ,  $\#\mathcal{A}$ . Let  $q_0, q_1$  be given natural numbers not exceeding  $N$  with  $(q_1, q_0 P(D_0)) = 1$  and  $\varphi(q_1) = q_1(1 + o(1))$ . Suppose that  $n \equiv a_0 \pmod{q_0}$  for all  $n \in \mathcal{A}$  with  $(a_0, q_0) = 1$ . An admissible set  $\mathcal{H}$  is given with

$$h \equiv 0 \pmod{q_0} \quad (h \in \mathcal{H})$$

and

$$(1.1) \quad p|h - h', \text{ with } h, h' \in \mathcal{H}, h \neq h', p > D_0 \text{ implies } p|q_0.$$

We now state “regularity conditions” on  $\mathcal{A}$ .

(I) We have

$$(1.2) \quad \sum_{\substack{q \leq N^\theta \\ (q, q_0 q_1) = 1}} \mu^2(q) \tau_{3k}(q) \left| \sum_{n \equiv a_q \bmod qq_0} X(\mathcal{A}; n) - \frac{Y}{qq_0} \right| \ll \frac{Y}{q_0 \mathcal{L}^{k+\varepsilon}}$$

(any  $a_q \equiv a_0 \pmod{q_0}$ ).

(II) There are nonnegative functions  $\varrho_1, \dots, \varrho_s$  defined on  $[N, 2N]$  (with  $s$  a constant,  $0 < a \leq s$ ) such that

$$(1.3) \quad X(\mathbb{P}; n) \geq \varrho_1(n) + \dots + \varrho_a(n) - (\varrho_{a+1}(n) + \dots + \varrho_s(n))$$

for  $n \in [N, 2N]$ . There are positive  $Y_{g,m}$  ( $g = 1, \dots, s$  and  $m = 1, \dots, k$ ) with

$$Y_{g,m} = Y(b_{g,m} + o(1)) \mathcal{L}^{-1}.$$

where the positive constants  $b_{g,m}$  satisfy

$$(1.4) \quad b_{1,m} + \dots + b_{a,m} - (b_{a+1,m} + \dots + b_{s,m}) \geq b > 0,$$

for  $m = 1, \dots, k$ . Moreover, for  $m \leq k$ ,  $g \leq s$  and any  $a_q \equiv a_0 \pmod{q_0}$  with  $(a_q, q) = 1$  defined for  $q \leq x^\theta$ ,  $(q, q_0 q_1) = 1$ , we have

$$(1.5) \quad \sum_{\substack{q \leq N^\theta \\ (q, q_0 q_1) = 1}} \mu^2(q) \tau_{3k}(q) \left| \sum_{n \equiv a_q \bmod qq_0} \varrho_g(n) X((\mathcal{A} + h_m) \cap \mathcal{A}; n) - \frac{Y_{g,m}}{\varphi(q_0 q)} \right| \ll \frac{Y}{\varphi(q_0) \mathcal{L}^{k+\varepsilon}}.$$

Finally,  $\varrho_g(n) = 0$  unless  $(n, P(N^{\theta/2})) = 1$ .

**Theorem 1.** *Under the above hypotheses on  $\mathcal{H}$  and  $\mathcal{A}$ , there is a set  $\mathcal{S}$  of  $t$  primes in  $\mathcal{A}$  with diameter not exceeding  $D(\mathcal{H})$ , provided that  $k \geq k_0(t, b, \theta)$  ( $k_0$  is defined at the end of this section).*

In proving Theorem 2, we shall take  $s = a = 1$ ,  $q_0 = q_1 = 1$ ,  $\rho_1(n) = X(\mathbb{P}; n)$ . A more complicated example with  $s = 5$ , of the inequality (1.3), occurs in proving Theorem 3, but again,  $q_0 = q_1 = 1$ . We shall consider elsewhere a result in which  $q_0, q_1$  are large. Maynard's Theorem 3.1 in [10] overlaps with our Theorem 1, but neither subsumes the other.

**Theorem 2.** *Let  $\alpha > 1$ ,  $\gamma = \alpha^{-1}$  and  $\beta \in \mathbb{R}$ . Suppose that*

$$(1.6) \quad \|\gamma r\| \gg r^{-3}$$

*for all  $r \in \mathbb{N}$ . Then for any  $N > c_1(t, \alpha, \beta)$ , there is a set of  $t$  primes of the form  $[\alpha m + \beta]$  in  $[N, 2N)$  having diameter*

$$< C_2 \alpha (\log \alpha + t) \exp(8t),$$

*where  $C_2$  is an absolute constant.*

**Theorem 3.** *Let  $\alpha$  be irrational with  $\alpha > 1$  and  $\beta \in \mathbb{R}$ . Let  $r \geq C_3(\alpha, \beta)$  and*

$$\left| \frac{1}{\alpha} - \frac{b}{r} \right| < \frac{1}{r^2}, \quad b \in \mathbb{N}, \quad (b, r) = 1.$$

*Let  $N = r^2$ . There is a set of  $t$  primes of the form  $[\alpha n + \beta]$  in  $[N, 2N)$  having diameter*

$$< C_4 \alpha (\log \alpha + t) \exp(7.743t),$$

*where  $C_4$  is an absolute constant.*

Theorem 3 improves Theorem 2 in that  $\alpha$  can be any irrational number in  $(1, \infty)$  and  $7.743 < 8$ , but we lose the arbitrary placement of  $N$ .

Turning our attention to our theorem of Bombieri-Vinogradov type, we write

$$E(N, N', \gamma, q, a) = \sup_I \left| \sum_{\substack{N \leq n < N' \\ \gamma n \in I \pmod{1} \\ n \equiv a \pmod{q}}} \Lambda(n) - \frac{(N' - N)|I|}{\varphi(q)} \right|.$$

Here,  $I$  runs over intervals of length  $|I| < 1$ .

**Theorem 4.** *Let  $A > 0$ ,  $\gamma$  be a real number and  $b/r$  a rational approximation to  $\gamma$ ,*

$$(1.7) \quad \left| \gamma - \frac{b}{r} \right| \leq \frac{1}{rN^{3/4}}, \quad N^\varepsilon \leq r \leq N^{3/4}, \quad (b, r) = 1.$$

*Then for  $N < N' \leq 2N$  and any  $A > 0$ , we have*

$$(1.8) \quad \sum_{q \leq \min(r, N^{1/4})N^{-\varepsilon}} \max_{(a, q)=1} E(N, N', \gamma, q, a) \ll N\mathcal{L}^{-A}.$$

Our Barban-Davenport-Halberstam type result is the following.

**Theorem 5.** *Let  $A > 0$  and  $\gamma$  be an irrational number. Suppose that for each  $\eta > 0$  and sufficiently large  $r \in \mathbb{N}$ , we have*

$$(1.9) \quad \|\gamma r\| > \exp(-r^\eta).$$

*Let  $N\mathcal{L}^{-A} \leq R \leq N$ . Then for  $N < N' \leq 2N$ ,*

$$(1.10) \quad \sum_{q \leq R} \sum_{\substack{a=1 \\ (a, q)=1}}^q E(N, N', \gamma, q, a)^2 \ll NR\mathcal{L}(\log \mathcal{L})^2.$$

There are weaker results overlapping with Theorems 4 and 5 by W. D. Banks and I. E. Shparlinski [4].

Let  $\gamma$  be irrational,  $\eta > 0$  and suppose that

$$\|\gamma r\| \leq \exp(-r^\eta)$$

for infinitely many  $r \in \mathbb{N}$ . Then (1.10) fails (so Theorem 5 is optimal in this sense). To see this, take  $N = \exp(r^{\eta/2})$ ,  $N' = 2N$ ,  $R = N\mathcal{L}^{-8/\eta}$ . We have, for some  $u \in \mathbb{Z}$ ,

$$\left| \gamma n - \frac{un}{r} \right| \leq 2Nr^{-1} \exp(-r^\eta) < \frac{1}{4r}, \quad (n \leq 2N).$$

From this, we infer that

$$\gamma n \notin \left( \frac{1}{4r}, \frac{3}{4r} \right) \pmod{1}, \quad (n \leq 2N).$$

So

$$E(N, 2N, \gamma, q, a)^2 \geq \frac{N^2}{4r^2 \varphi(q)}, \quad (q \leq R, (a, q) = 1).$$

Therefore,

$$\sum_{q \leq R} \sum_{\substack{a=1 \\ (a, q)=1}}^q E(N, 2N, \gamma, q, a)^2 \geq \frac{N^2}{4r^2} \sum_{q \leq R} \frac{1}{\varphi(q)} > \frac{N^2}{r^2} = NR\mathcal{L}^{4/\eta}.$$

We now turn to the definition of  $k_0(t, b, \theta)$ . For a smooth function  $F$  supported on

$$\mathcal{R}_k = \left\{ (x_1, \dots, x_k) \in [0, 1]^k : \sum_{i=1}^k x_i \leq 1 \right\},$$

set

$$I_k(F) = \int_0^1 \cdots \int_0^1 F(t_1, \dots, t_k)^2 dt_1 \cdots dt_k,$$

and

$$J_k^{(m)}(F) = \int_0^1 \cdots \int_0^1 \left( \int_0^1 F(t_1, \dots, t_k) dt_m \right)^2 dt_1 \cdots dt_{m-1} dt_{m+1} \cdots dt_k$$

for  $m = 1, \dots, k$ . Let

$$M_k = \sup_F \frac{\sum_{m=1}^k J_k^{(m)}(F)}{I_k(F)},$$

where the sup is taken over all functions  $F$  specified above and subject to the conditions  $I_k(F) \neq 0$  and  $J_k^{(m)}(F) \neq 0$  for each  $m$ . Sharpening a result of Maynard [9], D. H. J. Polymath [11] gives the lower bound

$$(1.11) \quad M_k \geq \log k + O(1).$$

Now let  $k_0(t, b, \theta)$  be the least integer  $k$  for which

$$(1.12) \quad M_k > \frac{2t-2}{b\theta}.$$

## 2. DEDUCTION OF THEOREM 1 FROM TWO PROPOSITIONS

We first write down some lemmas that we shall need later.

**Lemma 1.** *Let  $\kappa, A_1, A_2, L > 0$ . Suppose that  $\gamma$  is a multiplicative function satisfying*

$$0 \leq \frac{\gamma(p)}{p} \leq 1 - A_1$$

*for all prime  $p$  and*

$$-L \leq \sum_{w \leq p \leq z} \frac{\gamma(p) \log p}{p} - \kappa \log \frac{z}{w} \leq A_2$$

*for any  $w$  and  $z$  with  $2 \leq w \leq z$ . Let  $g$  be the totally multiplicative function defined by*

$$g(p) = \frac{\gamma(p)}{p - \gamma(p)}.$$

*Suppose that  $G : [0, 1] \rightarrow \mathbb{R}$  is a piecewise differentiable function with*

$$|G(y)| + |G'(y)| \leq B$$

*for  $0 \leq y \leq 1$  and*

$$(2.1) \quad S = \prod_p \left(1 - \frac{\gamma(p)}{p}\right)^{-1} \left(1 - \frac{1}{p}\right)^\kappa.$$

*Then for  $z > 1$ , we have*

$$\sum_{d < z} \mu(d)^2 g(d) G\left(\frac{\log d}{\log z}\right) = \frac{S(\log z)^\kappa}{\Gamma(\kappa)} \int_0^1 t^{\kappa-1} G(t) dt + O(SLB(\log z)^{\kappa-1}).$$

*The implied constant above depends on  $A_1, A_2, \kappa$ , but is independent of  $L$ .*

*Proof.* This is [7, Lemma 4]. □

Throughout this section, we assume that the hypotheses of Theorem 1 hold. Moreover, we write

$$W_1 = \prod_{p \leq D_0 \text{ or } p|q_0 q_1} p, \quad W_2 = \prod_{\substack{p \leq D_0 \\ p \nmid q_0}} p, \quad R = N^{\theta/2-\varepsilon}.$$

Recalling the definition of admissible set, we pick a natural number  $\nu_0$  with

$$(\nu_0 + h_m, W_2) = 1 \quad (m = 1, \dots, k).$$

**Lemma 2.** *Suppose that  $\gamma(p) = 1 + O(p^{-1})$  if  $p \nmid W_1$  and  $\gamma(p) = 0$  if  $p|W_1$ . Let  $\kappa = 1$  and  $S$  as defined in (2.1). We have*

$$S = \frac{\varphi(W_1)}{W_1} (1 + O(D_0^{-1})).$$

*Proof.* We have

$$S = \prod_{p|W_1} \left(1 - \frac{1}{p}\right) \prod_{p \nmid W_1} \left(1 - \frac{1}{p} + O\left(\frac{1}{p^2}\right)\right)^{-1} \left(1 - \frac{1}{p}\right) = \frac{\varphi(W_1)}{W_1} \prod_{\substack{p > D_0 \\ p \nmid q_0 q_1}} (1 + O(p^{-2})),$$

from which the statement of the lemma can be readily obtained. □

**Lemma 3.** *Let  $H > 1$ ,*

$$T_1 = \sum_{\substack{d \leq R \\ (d, W_1)=1}} \frac{\mu^2(d)}{d} \sum_{a|d} \frac{4^{\omega(a)}}{a} \quad \text{and} \quad T_2 = \sum_{H < d \leq R} \frac{\mu^2(d)}{d^2} \sum_{a|d} a^{-1/2}.$$

Then, we have

$$(2.2) \quad T_1 \ll \frac{\varphi(W_1)}{W_1} \mathcal{L}$$

and

$$(2.3) \quad T_2 \ll H^{-1}.$$

*Proof.* Let  $\gamma(p) = 0$  if  $p|W_1$  and

$$\gamma(p) = \frac{p^2 + 4p}{p^2 + p + 4}$$

if  $p \nmid W_1$ . Then  $g(p)$ , as defined in the statement of Lemma 1, is

$$g(p) = \frac{1}{p} + \frac{4}{p^2}$$

if  $p \nmid W_1$ . Therefore, if  $d$  is square-free and  $(d, W_1) = 1$ ,

$$\frac{1}{d} \sum_{a|d} \frac{4^{\omega(a)}}{a} = \frac{1}{d} \prod_{p|d} \left(1 + \frac{4}{p}\right) = g(d).$$

Otherwise, if  $(d, W_1) \neq 1$ , then  $g(d) = 0$ . Using Lemma 1 with  $G(y) = 1$  and Lemma 2, we have

$$T_1 = \sum_{d \leq R} \mu^2(d) g(d) G\left(\frac{\log d}{\log R}\right) = \frac{\varphi(W_1)}{W_1} (1 + O(D_0^{-1})) \log R + O\left(\frac{\varphi(W_1)}{W_1} L\right),$$

where we can take

$$L = \sum_{p|W_1} \frac{\log p}{p} \ll \log D_0 + \log \omega(q_0) \ll \log \mathcal{L}.$$

Combining everything, we get (2.2).

To prove (2.3), we interchange the summations and get

$$T_2 \leq \sum_{a \leq R} a^{-5/2} \sum_{Ha^{-1} < k \leq Ra^{-1}} k^{-2} \ll \sum_{a \leq R} a^{-3/2} H^{-1} \ll H^{-1},$$

completing the proof of the lemma.  $\square$

**Lemma 4.** Let  $f_0, f_1$  be multiplicative functions with  $f_0(p) = f_1(p) + 1$ . Then for squarefree  $d, e$ ,

$$\frac{1}{f_0([d, e])} = \frac{1}{f_0(d)f_0(e)} \sum_{k|d, e} f_1(k).$$

*Proof.* We have

$$\frac{1}{f_0(d)f_0(e)} \sum_{k|d, e} f_1(k) = \frac{1}{f_0(d)f_0(e)} \prod_{p|(d, e)} (1 + f_1(p)) = \frac{1}{f_0(d)f_0(e)} \prod_{p|(d, e)} f_0(p) = \prod_{p|[d, e]} (f_0(p))^{-1}.$$

The lemma follows from this.  $\square$

We now prove two propositions that readily yield Theorem 1 when combined. To state them, we define weights  $y_{\mathbf{r}}$  and  $\lambda_{\mathbf{r}}$  for tuples

$$\mathbf{r} = (r_1, \dots, r_k) \in \mathbb{N}^k$$

having the properties

$$(2.4) \quad \left(\prod_{i=1}^k r_i, W_1\right) = 1, \quad \mu^2\left(\prod_{i=1}^k r_i\right) = 1.$$

We set  $y_{\mathbf{r}} = \lambda_{\mathbf{r}} = 0$  for all other tuples. Let  $F$  be a smooth function with  $|F| \leq 1$  and the properties given at the end of Section 1. Let

$$(2.5) \quad y_{\mathbf{r}} = F\left(\frac{\log r_1}{\log R}, \dots, \frac{\log r_k}{\log R}\right),$$

and

$$(2.6) \quad \lambda_{\mathbf{d}} = \prod_{i=1}^k \mu(d_i) d_i \sum_{\substack{\mathbf{r} \\ d_i | r_i \forall i}} \frac{y_{\mathbf{r}}}{\prod_{i=1}^k \varphi(r_i)}.$$

We have

$$(2.7) \quad \lambda_{\mathbf{r}} \ll \mathcal{L}^k$$

(see (5.9) of [9]). For  $n \equiv \nu_0 \pmod{W_2}$ , let

$$w_n = \left( \sum_{d_i | n + h_i \forall i} \lambda_{\mathbf{d}} \right)^2,$$

and  $w_n = 0$  for all other natural numbers  $n$ .

**Proposition 1.** *Let*

$$S_1 = \sum_{N \leq n < 2N} w_n X(\mathcal{A}; n).$$

*Then*

$$S_1 = \frac{(1 + o(1)) \varphi(W_1)^k Y(\log R)^k I_k(F)}{q_0 W_1^k W_2}.$$

**Proposition 2.** *Let*

$$S_2(g, m) = \sum_{\substack{N \leq n < 2N \\ n \in \mathcal{A} \cap (\mathcal{A} - h_m)}} w_n \varrho_g(n + h_m).$$

*Then for  $1 \leq g \leq s$  and  $1 \leq m \leq k$ ,*

$$S_2(g, m) = \frac{b_{g,m} (1 + o(1)) \varphi(W_1)^{k+1} Y(\log R)^{k+1} J_k^{(m)}(F)}{\varphi(q_0) \varphi(W_2) W_1^{k+1} \mathcal{L}}.$$

Before proving the above propositions, we shall deduce Theorem 1 from them.

*Proof of Theorem 1.* Let

$$Z = \frac{Y \varphi(W_1)^k}{q_0 W_1^k W_2} (\log R)^k$$

and

$$S(N) = \sum_{\substack{N \leq n < 2N \\ n \in \mathcal{A}}} w_n \left( \sum_{m=1}^k X(\mathbb{P} \cap \mathcal{A}; n + h_m) - (t-1) \right).$$

Since  $w_n \geq 0$ , (1.3) gives that

$$S(N) \geq \sum_{m=1}^k \left( \sum_{g=1}^a S_2(g, m) - \sum_{g=a+1}^s S_2(g, m) \right) - (t-1) S_1.$$

Using Propositions 1 and 2, the right-hand side of the above is

$$(1 + o(1)) Z \left( \sum_{m=1}^k \left( \sum_{g=1}^a b_{g,m} - \sum_{g=a+1}^s b_{g,m} \right) J_k^{(m)}(F) \left( \frac{\theta}{2} - \varepsilon \right) - (t-1) I_k(F) \right).$$

Here we have used

$$\frac{\varphi(q_0)\varphi(q_1)\varphi(W_2)}{q_0q_1W_2} \frac{W_1}{\varphi(W_1)} = 1 \quad \text{and} \quad \frac{\varphi(q_1)}{q_1} = 1 + o(1).$$

Therefore, using (1.4), we get

$$S(N) \geq (1 + o(1)) Z \left( b \sum_{m=1}^k J_k^{(m)}(F) \left( \frac{\theta}{2} - \varepsilon \right) - (t-1)I_k(F) \right) > 0,$$

for a suitable choice of  $F$ . The positivity of the above expression is a consequence of (1.12). Therefore, there must be at least one  $n \in \mathcal{A}$  for which

$$\sum_{m=1}^k X(\mathbb{P} \cap \mathcal{A}; n + h_m) > t - 1.$$

For this  $n$ , there is a set of  $t$  primes  $n + h_{m_1}, \dots, n + h_{m_t}$  in  $\mathcal{A}$ . □

### 3. PROOF OF PROPOSITIONS 1 AND 2

This section is devoted to the proofs of the two propositions.

*Proof of Proposition 1.* We first show that

$$(3.1) \quad S_1 = \frac{Y}{q_0 W_2} \sum_{\mathbf{r}} \frac{y_{\mathbf{r}}^2}{\prod_{i=1}^k \varphi(r_i)} + O\left(\frac{Y \varphi(W_1)^k \mathcal{L}^k}{q_0 W_2 W_1^k D_0}\right).$$

From the definition of  $w_n$ , we get

$$(3.2) \quad S_1 = \sum_{\mathbf{d}, \mathbf{e}} \lambda_{\mathbf{d}} \lambda_{\mathbf{e}} \sum_{\substack{N \leq n < 2N \\ n \equiv \nu_0 \pmod{W_2} \\ [d_i, e_i] | n + h_i \ \forall i}} X(\mathcal{A}; n).$$

Recall that  $n \equiv a_0 \pmod{q_0}$  for all  $n \in \mathcal{A}$ . The inner sum of the above takes the form

$$\sum_{\substack{N \leq n < 2N \\ n \equiv a_q \pmod{qq_0}}} X(\mathcal{A}; n),$$

where

$$q = W_2 \prod_{i=1}^k [d_i, e_i],$$

provided that  $W_2, [d_1, e_1], \dots, [d_k, e_k]$  are pairwise coprime. The latter restriction reduces to

$$(3.3) \quad (d_i, e_j) = 1$$

for all  $i \neq j$ , and we exhibit this condition on the summation by writing

$$\sum'_{\mathbf{d}, \mathbf{e}}.$$

Outside of  $\sum'_{\mathbf{d}, \mathbf{e}}$ , the inner sum is empty. To see this, suppose that  $p|d_i, p|e_j$  with  $i \neq j$ , then the conditions

$$[d_i, e_i] | n + h_i, \text{ and } [d_j, e_j] | n + h_j$$

imply that  $p|h_i - h_j$ . This means that either  $p \leq D_0$  or  $p|q_0$ , both contrary to  $p|d_i$ .

Counting the number of times a given  $q$  can arise, we get

$$(3.4) \quad S_1 - \frac{Y}{q_0 W_2} \sum'_{\mathbf{d}, \mathbf{e}} \frac{\lambda_{\mathbf{d}} \lambda_{\mathbf{e}}}{\prod_{i=1}^k [d_i, e_i]} \ll \left( \max_{\mathbf{d}} |\lambda_{\mathbf{d}}| \right)^2 \sum_{\substack{q \leq R^2 W_2 \\ (q, q_0) = 1}} \mu^2(q) \tau_{3k}(q) \left| \sum_{n \equiv a_q \pmod{qq_0}} X(\mathcal{A}; n) - \frac{Y}{qq_0} \right|.$$



Since  $R^2 W_2 \leq N^\theta$ , we can appeal to (1.2) and (2.7) to majorize the right-hand side of (3.4) by

$$\ll \frac{Y}{q_0} \mathcal{L}^{2k-(k+\varepsilon)} \ll \frac{\varphi(W_1)^k Y \mathcal{L}^k}{q_0 W_2 W_1^k D_0}.$$

Applying Lemma 4 with  $f_1 = \varphi$ , we see that

$$S_1 = \frac{Y}{q_0 W_2} \sum_{\mathbf{u}} \prod_{i=1}^k \varphi(u_i) \sum'_{\substack{\mathbf{d}, \mathbf{e} \\ u_i | d_i, e_i \ \forall i}} \frac{\lambda_{\mathbf{d}} \lambda_{\mathbf{e}}}{\prod_{i=1}^k d_i e_i} + O\left(\frac{\varphi(W_1)^k Y \mathcal{L}^k}{q_0 W_2 W_1^k D_0}\right).$$

Now we follow [9] verbatim to transform this equation into

$$(3.5) \quad S_1 = \frac{Y}{q_0 W_2} \sum_{\mathbf{u}} \prod_{i=1}^k \varphi(u_i) \sum_{s_{1,2}, \dots, s_{k,k-1}}^* \prod_{\substack{1 \leq i, j \leq k \\ i \neq j}} \mu(s_{i,j}) \sum_{\substack{\mathbf{d}, \mathbf{e} \\ u_i | d_i, e_i \ \forall i \\ s_{i,j} | d_i, e_j \ \forall i \neq j}} \frac{\lambda_{\mathbf{d}} \lambda_{\mathbf{e}}}{\prod_{i=1}^k d_i e_i} + O\left(\frac{\varphi(W_1)^k Y \mathcal{L}^k}{q_0 W_2 W_1^k D_0}\right).$$

Here  $\sum^*$  indicates that  $(s_{i,j}, u_i u_j) = 1$  and  $(s_{i,j}, s_{i,c}) = 1 = (s_{i,j}, s_{d,j})$ , for  $c \neq j$ ,  $d \neq i$ . Now define

$$(3.6) \quad a_j = u_j \prod_{i \neq j} s_{j,i}, \quad b_j = u_j \prod_{i \neq j} s_{i,j}.$$

As in [9], we recast (3.5) as

$$(3.7) \quad S_1 = \frac{Y}{q_0 W_2} \sum_{\mathbf{u}} \prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} \sum_{s_{1,2}, \dots, s_{k,k-1}}^* \prod_{\substack{1 \leq i, j \leq k \\ i \neq j}} \mu(s_{i,j}) \sum_{\substack{\mathbf{d}, \mathbf{e} \\ u_i | d_i, e_i \ \forall i \\ s_{i,j} | d_i, e_j \ \forall i \neq j}} \frac{\mu(s_{i,j})}{\varphi(s_{i,j})^2} y_{\mathbf{a}} y_{\mathbf{b}} + O\left(\frac{\varphi(W_1)^k Y \mathcal{L}^k}{q_0 W_2 W_1^k D_0}\right).$$

For the non-zero terms on the right-hand side of (3.7), either  $s_{i,j} = 1$  or  $s_{i,j} > D_0$ . The terms of the latter kind (for given  $i, j$ ,  $i \neq j$ ) contribute

$$(3.8) \quad \ll \frac{Y}{q_0 W_2} \left( \sum_{\substack{u \leq R \\ (u, W_1)=1}} \frac{\mu(u)^2}{\varphi(u)} \right)^k \left( \sum_{s_{i,j} > D_0} \frac{\mu(s_{i,j})^2}{\varphi(s_{i,j})^2} \right) \left( \sum_{s \geq 1} \frac{\mu(s)}{\varphi(s)^2} \right)^{k^2-k-1} = \frac{Y}{q_0 W_2} U_1 U_2 U_3,$$

say. Clearly,  $U_3 \ll 1$ . Now if  $u$  is squarefree, we have

$$\frac{1}{\varphi(u)} = \frac{1}{u} \prod_{p|u} \left(1 - \frac{1}{p}\right)^{-1} \ll \frac{1}{u} \sum_{a|u} \frac{1}{a}$$

and

$$\frac{1}{\varphi(u)^2} \ll \frac{1}{u^2} \prod_{p|u} \left(1 + \frac{2}{p}\right) = \frac{1}{u^2} \sum_{a|u} \frac{2^{\omega(a)}}{a} \ll \frac{1}{u^2} \sum_{a|u} a^{-1/2}.$$

So (2.2) and (2.3) give, respectively,

$$U_1 \ll \left( \frac{\varphi(W_1)}{W_1} \mathcal{L} \right)^k \quad \text{and} \quad U_2 \ll \frac{1}{D_0}.$$

Hence, the right-hand side of (3.8) is

$$\ll \frac{\varphi(W_1)^k Y \mathcal{L}^k}{q_0 W_2 W_1^k D_0}$$

and we have (3.1).

Now, we shall deduce Proposition 1 from (3.1). Mindful of (2.6), we have

$$S_1 = \frac{Y}{q_0 W_2} \sum_{\substack{\mathbf{u} \\ (u_l, u_j)=1 \forall l \neq j \\ (u_l, W_1)=1 \forall l}} \prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} F\left(\frac{\log u_1}{\log R}, \dots, \frac{\log u_k}{\log R}\right)^2 + O\left(\frac{\varphi(W_1)^k Y \mathcal{L}^k}{q_0 W_2 W_1^k D_0}\right).$$

Note that the common prime factors of two integers both coprime to  $W_1$  are strictly greater than  $D_0$ . Thus, we may drop the condition  $(u_l, u_j) = 1$  in the above expression at the cost of an error of size

$$\ll \frac{Y}{q_0 W_2} \sum_{p > D_0} \sum_{\substack{u_1 \dots u_k < R \\ p | u_l, u_j \\ (u_l, W_1)=1 \forall l}} \prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} \ll \frac{Y}{q_0 W_2} \sum_{p > D_0} \frac{1}{(p-1)^2} \left( \sum_{\substack{u \leq R \\ (u, W_1)=1}} \frac{\mu(u)^2}{\varphi(u)} \right)^k \ll \frac{\varphi(W_1)^k Y \mathcal{L}^k}{q_0 W_2 W_1^k D_0},$$

by virtue of (2.2).

It remains to evaluate the sum

$$(3.9) \quad \sum_{\substack{\mathbf{u} \\ (u_l, W_1)=1 \forall l}} \prod_{i=1}^k \frac{\mu(u_i)^2}{\varphi(u_i)} F\left(\frac{\log u_1}{\log R}, \dots, \frac{\log u_k}{\log R}\right)^2.$$

This requires applying Lemma 1  $k$  times with

$$\gamma(p) = \begin{cases} 0 & p | W_1, \\ 1 & p \nmid W_1. \end{cases}$$

We take  $A_1$  and  $A_2$  to be suitable constants and

$$L \ll 1 + \sum_{p | W_1} \frac{\log p}{p} \ll \log \mathcal{L}$$

as noted earlier. In the  $j$ -th application, we replace the summation over  $u_j$  by the integral over  $[0, 1]$ . Ultimately, we express the sum in (3.9) in the form

$$\frac{\varphi(W_1)^k}{W_1^k} (\log R)^k I_k(F) + O\left(\frac{\varphi(W_1)(\log \mathcal{L}) \mathcal{L}^{k-1}}{W_1^k}\right)$$

and Proposition 1 follows at once.  $\square$

We shall need the following lemma in the proof of Proposition 2.

**Lemma 5.** *Let  $1 \leq m \leq k$  and suppose that  $r_m = 1$ . Let*

$$y_{\mathbf{r}}^{(m)} = \prod_{i=1}^k \mu(r_i) g(r_i) \sum_{\substack{\mathbf{d} \\ r_i | d_i \forall i \\ d_m = 1}} \frac{\lambda_{\mathbf{d}}}{\prod_{i=1}^k \varphi(d_i)}.$$

Then

$$y_{\mathbf{r}}^{(m)} = \sum_{a_m} \frac{y_{r_1, \dots, r_{m-1}, a_m, r_{m+1}, \dots, r_k}}{\varphi(a_m)} + O\left(\frac{\varphi(W_1) \mathcal{L}}{W_1 D_0}\right).$$

*Proof.* Following [9] verbatim, we have

$$(3.10) \quad y_{\mathbf{r}}^{(m)} = \prod_{i=1}^k \mu(r_i) g(r_i) \sum_{\substack{\mathbf{a} \\ r_i | a_i \forall i}} \frac{y_{\mathbf{a}}}{\prod_{i=1}^k \varphi(a_i)} \prod_{i \neq m} \frac{\mu(a_i) r_i}{\varphi(a_i)}.$$

Fix  $j$ ,  $1 \leq j \leq k$ . In (3.10), the nonzero terms will have either  $a_j = r_j$  or  $a_j > D_0 r_j$ . The contribution from the terms with  $a_j \neq r_j$  is

$$(3.11) \quad \ll \prod_{i=1}^k g(r_i) r_i \left( \sum_{\substack{a_j > D_0 r_j \\ r_j | a_j}} \frac{\mu(a_j)^2}{\varphi(a_j)^2} \right) \left( \sum_{\substack{a_m \leq R \\ (a_m, W_1)=1}} \frac{\mu(a_m)^2}{\varphi(a_m)} \right) \prod_{\substack{1 \leq i \leq k \\ i \neq j, m}} \sum_{r_i | a_i} \frac{\mu(a_i)^2}{\varphi(a_i)^2}.$$

Now, as before, from (2.2) and (2.3),

$$\sum_{\substack{a_j > D_0 r_j \\ r_j | a_j}} \frac{\mu(a_j)^2}{\varphi(a_j)^2} \ll \frac{1}{D_0 \varphi(r_j)^2}, \quad \sum_{\substack{a_m \leq R \\ (a_m, W_1)=1}} \frac{\mu(a_m)^2}{\varphi(a_m)} \ll \frac{\varphi(W_1)}{W_1} \mathcal{L}$$

and

$$\sum_{r_i | a_i} \frac{\mu(a_i)^2}{\varphi(a_i)^2} \leq \frac{\mu(r_i)^2}{\varphi(r_i)^2} \sum_k \frac{\mu(k)}{\varphi(k)^2} \ll \frac{1}{\varphi(r_i)^2},$$

majorizing (3.11) by

$$\ll \prod_{i=1}^k \frac{g(r_i) r_i}{\varphi(r_i)^2} \frac{\varphi(W_1)}{W_1 D_0} \mathcal{L} \ll \frac{\varphi(W_1) \mathcal{L}}{W_1 D_0}.$$

Hence (3.10) becomes

$$y_{\mathbf{r}}^{(m)} = \prod_{i=1}^k \frac{g(r_i) r_i}{\varphi(r_i)^2} \sum_{a_m} \frac{y_{r_1, \dots, r_{m-1}, a_m, r_{m+1}, \dots, r_k}}{\varphi(a_m)} + O\left(\frac{\varphi(W_1) \mathcal{L}}{W_1 D_0}\right),$$

and the proof is completed by applying Lemma 2.  $\square$

Now we proceed to the proof of Proposition 2.

*Proof of Proposition 2.* Let

$$y_{\max}^{(m)} = \max_{\mathbf{r}} |y_{\mathbf{r}}^{(m)}|,$$

where  $y_{\mathbf{r}}^{(m)}$  is defined in Lemma 5. We shall first show that

$$(3.12) \quad S_2(g, m) = \frac{Y_{g,m}}{\varphi(q_0) \varphi(W_2)} \sum_{\mathbf{u}} \frac{\left(y_{\mathbf{u}}^{(m)}\right)^2}{\prod_{i=1}^k g(u_i)} + O\left(\frac{Y \mathcal{L}^{k-2} \varphi^{k-1}(W_1) \left(y_{\max}^{(m)}\right)^2}{\varphi(q_0) \varphi(W_2) W_1^{k-1} D_0} + \frac{Y \mathcal{L}^{k-\varepsilon}}{\varphi(q_0)}\right).$$

From the definition of  $w_n$ , we have

$$(3.13) \quad S_2(g, m) = \sum_{\mathbf{d}, \mathbf{e}} \lambda_{\mathbf{d}} \lambda_{\mathbf{e}} \sum_{\substack{n \in \mathcal{A} \cap (\mathcal{A} - h_m) \\ N \leq n < 2N, n \equiv \nu_0 \pmod{W_2} \\ [d_i, e_i] | n + h_i \ \forall i}} \varrho_g(n + h_m).$$

As in the proof of Proposition 1,  $\sum_{\mathbf{d}, \mathbf{e}}$  reduces to  $\sum'_{\mathbf{d}, \mathbf{e}}$ . Let  $n' = n + h_m$ . Since  $n + h_m \equiv a_0 \pmod{q_0}$  for  $n \in \mathcal{A}$ , the inner sum of (3.13) reduces to

$$T(\mathbf{d}, \mathbf{e}) := \sum_{\substack{n' \equiv \nu_0 + h_m \pmod{W_2} \\ n' \equiv a_0 \pmod{q_0} \\ n' \equiv h_m - h_i \pmod{[d_i, e_i] \forall i}} X(\mathcal{A} \cap (\mathcal{A} + h_m), n') \varrho_g(n').$$

Recall that  $\varrho_g(n') = 0$  if  $n'$  is divisible by a prime divisor of  $[d_i, e_i]$ . Since one condition of the summation is  $[d_m, e_m] | n'$ , we have  $T(\mathbf{d}, \mathbf{e}) = 0$  unless  $d_m = e_m = 1$ . When  $d_m = e_m = 1$ ,

$$T(\mathbf{d}, \mathbf{e}) = \sum_{n \equiv a_q \pmod{q q_0}} X(\mathcal{A} \cap (\mathcal{A} + h_m), n) \varrho_g(n).$$

Here we have

$$q = W_2 \prod_{i=1}^k [d_i, e_i], \quad (a_q, q) = 1, \quad a_q \equiv a_0 \pmod{q_0}.$$

For  $(a_q, q) = 1$ , we need  $(h_m - h_i, [d_i, e_i]) = 1$  whenever  $m \neq i$ , which was noted earlier.

Arguing as in the proof of Proposition 1, (1.5) now gives

$$S_2(g, m) = \frac{Y_{g,m}}{\varphi(q_0)\varphi(W_2)} \sum'_{\substack{\mathbf{d}, \mathbf{e} \\ d_m = e_m = 1}} \frac{\lambda_{\mathbf{d}} \lambda_{\mathbf{e}}}{\prod_{i=1}^k \varphi([d_i, e_i])} + O\left(\frac{Y \mathcal{L}^{k-\varepsilon}}{\varphi(q_0)}\right).$$

With  $a_j$  and  $b_j$  as in (3.6), we follow [9] to obtain

$$(3.14) \quad S_2(g, m) = \frac{Y_{g,m}}{\varphi(q_0)\varphi(W_2)} \sum_{\mathbf{u}} \prod_{i=1}^k \frac{\mu(u_i)^2}{g(u_i)} \sum_{s_{1,2}, \dots, s_{k,k-1}}^* \prod_{\substack{1 \leq i,j \leq k \\ i \neq j}} \frac{\mu(s_{i,j})}{g^2(s_{i,j})} y_{\mathbf{a}}^{(m)} y_{\mathbf{b}}^{(m)} + O\left(\frac{Y \mathcal{L}^{k-\varepsilon}}{\varphi(q_0)}\right).$$

Here  $q$  is the totally multiplicative function with  $g(p) = p-2$  for all  $p$  and we have used Lemma 4 with  $f_1 = g$ .

The contribution to the sum in (3.14) from  $s_{i,j} \neq 1$  (for given  $i, j$ ) is

$$(3.15) \quad \begin{aligned} & \ll \frac{Y \left(y_{\max}^{(m)}\right)^2}{\varphi(q_0)\varphi(W_2)\mathcal{L}} \left( \sum_{\substack{u < R \\ (u, W_1)=1}} \frac{\mu(u)^2}{g(u)} \right)^{k-1} \left( \sum_s \frac{\mu(s)^2}{g(s)^2} \right)^{k(k-1)-1} \left( \sum_{s_{i,j} > D_0} \frac{\mu(s_{i,j})^2}{g(s_{i,j})^2} \right) \\ & = \frac{Y \left(y_{\max}^{(m)}\right)^2}{\varphi(q_0)\varphi(W_2)\mathcal{L}} V_1 V_2 V_3, \end{aligned}$$

say. Clearly,  $V_2 \ll 1$ . Using (2.2) while mindful of the estimate

$$\frac{1}{g(s)} \ll \frac{1}{s} \sum_{a|s} \frac{2^{\omega(a)}}{a}$$

yields that

$$V_1 \ll \left( \frac{\varphi(W_1)}{W_1} \mathcal{L} \right)^{k-1}.$$

From (2.3) and the observation that, for  $s$  squarefree,

$$\frac{1}{g^2(s)} \ll \frac{1}{s^2} \sum_{a|s} \frac{4^{\omega(a)}}{a} \ll \frac{1}{s^2} \sum_{a|s} a^{-1/2},$$

we get that

$$V_3 \ll D_0^{-1}.$$

Note the bound in (3.15) is

$$\ll \frac{Y \left(y_{\max}^{(m)}\right)^2 \mathcal{L}^{k-2}}{\varphi(q_0)\varphi(W_2)} \left( \frac{\varphi(W_1)}{W_1} \right)^{k-1} \frac{1}{D_0},$$

and we have established (3.12).

Now we use Lemma 5 in (3.12), recalling (2.5). When  $r_m = 1$ ,

$$(3.16) \quad y_{\mathbf{r}}^{(m)} = \sum_{(u, W_1 \prod_{i=1}^k r_i)=1} \frac{\mu(u)^2}{\varphi(u)} F\left(\frac{\log r_1}{\log R}, \dots, \frac{\log r_{m-1}}{\log R}, \frac{\log u}{\log R}, \frac{\log r_{m+1}}{\log R}, \dots, \frac{\log r_k}{\log R}\right) + O\left(\frac{\varphi(W_1)\mathcal{L}}{W_1 D_0}\right).$$

From this, we find that

$$y_{\max}^{(m)} \ll \frac{\varphi(W_1)}{W_1} \mathcal{L}.$$

We shall apply Lemma 1 to (3.16) with  $\kappa = 1$ ,

$$\gamma(p) = \begin{cases} 1, & p \nmid W_1 \prod_{i=1}^k r_i \\ 0, & \text{otherwise,} \end{cases}$$

$A_1, A_2$  suitably chosen and

$$L \ll \log \mathcal{L}$$

(similar to the proof of (2.2)). Define

$$F_{\mathbf{r}}^{(m)} = \int_0^1 F\left(\frac{\log r_1}{\log R}, \dots, \frac{\log r_{m-1}}{\log R}, t_m, \frac{\log r_{m+1}}{\log R}, \dots, \frac{\log r_k}{\log R}\right) dt_m.$$

We obtain that

$$y_{\mathbf{r}}^{(m)} = \log R \frac{\varphi(W_1)}{W_1} \left( \prod_{i=1}^k \frac{\varphi(r_i)}{r_i} \right) F_{\mathbf{r}}^{(m)} + O\left(\frac{\varphi(W_1)\mathcal{L}}{W_1 D_0}\right).$$

Inserted into (3.12), the above produces a main term

$$(3.17) \quad \frac{(\log R)^2 Y_{g,m} \varphi(W_1)^2}{\varphi(q_0) \varphi(W_2) W_1^2} \sum_{\substack{\mathbf{r} \\ (r_i, W_1)=1 \forall i \\ (r_i, r_j)=1 \forall i \neq j \\ r_m=1}} \prod_{i=1}^k \frac{\varphi(r_i) \mu(r_i)^2}{g(r_i) r_i^2} \left( F_{\mathbf{r}}^{(m)} \right)^2$$

and an error term of size

$$\ll \frac{Y_{g,m}}{\varphi(q_0) \varphi(W_2)} \sum_{r_m=1} \frac{\varphi(W_1)^2 \mathcal{L}^2}{W_1^2 D_0 \prod_{i=1}^k g(r_i)} \ll \frac{Y \varphi(W_1)^2 \mathcal{L}^2}{\varphi(q_0) \varphi(W_2) W_1^2 D_0} \left( \sum_{\substack{r < R \\ (r, W_1)=1}} \frac{1}{g(r)} \right)^{k-1} \ll \frac{Y \varphi(W_1)^{k+1} \mathcal{L}^k}{\varphi(q_0) \varphi(W_2) W_1^{k+1} D_0}.$$

Recall that  $Y_{g,m} \ll Y \mathcal{L}^{-1}$ . Now we remove the condition  $(r_i, r_j) = 1$  from (3.17). As before, this introduces an error of size

$$\ll \frac{\mathcal{L}^2 Y \varphi(W_1)^2}{\varphi(q_0) \varphi(W_2) W_1^2} \left( \sum_{p > D_0} \frac{\varphi(p)^2}{g(p)^2 p^2} \right) \left( \sum_{\substack{r < R \\ (r, W_1)=1}} \frac{\mu(r)^2 \varphi(r)}{g(r) r} \right)^{k-1} \ll \frac{Y \mathcal{L}^k \varphi(W_1)^{k+1}}{\varphi(q_0) \varphi(W_2) W_1^{k+1} D_0}$$

by an application of Lemma 3. Combining all our results, we get

$$S_2(g, m) = \frac{(\log R)^2 Y_{g,m} \varphi(W_1)^2}{\varphi(q_0) \varphi(W_2) W_1^2} \sum_{\substack{\mathbf{r} \\ (r_i, W_1)=1 \forall i \\ r_m=1}} \prod_{i=1}^k \frac{\varphi(r_i)^2 \mu(r_i)^2}{g(r_i) r_i^2} \left( F_{\mathbf{r}}^{(m)} \right)^2 + O\left(\frac{Y \varphi(W_1)^{k+1} \mathcal{L}^k}{\varphi(q_0) \varphi(W_2) W_1^{k+1} D_0}\right).$$

The last sum is evaluated by applying Lemma 1 to each summation variable in turn, taking

$$\gamma(p) = \begin{cases} \frac{p^3 - 2p^2 + p}{p^3 - p^2 - 2p + 1}, & p \nmid W_1 \\ 0, & p | W_1 \end{cases}$$

to produce the right value of  $\gamma(p)/(p - \gamma(p))$ . Of course

$$S = \frac{\varphi(W_1)}{W_1} (1 + O(D_0^{-1}))$$

by Lemma 2, while  $L \ll \log \mathcal{L}$ . Our final conclusion is that

$$S_2(g, m) = \frac{(\log R)^{k+1} Y_{g,m} \varphi(W_1)^{k+1} J_k^{(m)}}{\varphi(q_0) \varphi(W_2) W_1^{k+1}} (1 + o(1))$$

completing the proof.  $\square$

#### 4. FURTHER LEMMAS

Let  $\gamma = \alpha^{-1}$ . As noted in [4], the set of  $[\alpha m + \beta]$  in  $[N, 2N)$  may be written as

$$\{n \in [N, 2N) : \gamma n \in (\gamma\beta - \gamma, \beta\gamma] \pmod{1}\}.$$

**Lemma 6.** *Let  $I = (a, b)$  be an interval of length  $l$  with  $0 < l < 1$  and let  $h$  be a natural number satisfying*

$$0 < -h\gamma < 2\varepsilon \pmod{1},$$

where  $2\varepsilon < l$ . Let

$$\mathcal{A} = \{n \in [N, 2N) : \gamma n \in I \pmod{1}\}.$$

Then

$$\mathcal{A} \cap (\mathcal{A} + h) = \{n \in [N + h, 2N) : \gamma n \in J \pmod{1}\}$$

where  $J$  is an interval of length  $l'$  with

$$l - 2\varepsilon < l' < l.$$

*Proof.* Let  $t \equiv -h\gamma \pmod{1}$ ,  $0 < t < 2\varepsilon$ . Clearly  $\mathcal{A} \cap (\mathcal{A} + h)$  consists of the integers in  $[N + h, 2N)$  for which

$$\gamma n \in (a, b) \pmod{1}, \gamma n + t \in (a, b) \pmod{1}.$$

The lemma follows with  $J = (a, b - t)$ .  $\square$

**Lemma 7.** *Let  $I$  be an interval of length  $l$ ,  $0 < l < 1$ . Let  $x_1, \dots, x_N$  be real. Then*

(i) *There exists  $z$  such that*

$$\#\{j \leq N : x_j \in z + I \pmod{1}\} \geq Nl.$$

(ii) *We have (for  $a_j \geq 0$ ,  $j = 1, \dots, N$  and  $L \geq 1$ )*

$$\sum_{\substack{j=1 \\ x_j \in I \pmod{1}}}^N a_j - l \sum_{j=1}^N a_j \ll L^{-1} \sum_{j=1}^N a_j + \sum_{h=1}^L h^{-1} \left| \sum_{j=1}^N a_j e(hx_j) \right|.$$

*Proof.* We leave (1) as an exercise; (2) is a slight variant of [1, Theorem 2.1].  $\square$

**Lemma 8.** *Let  $1 \leq Q \leq N$  and  $F$  a nonnegative function defined on Dirichlet characters. Then for some  $Q_1$ ,  $1 \leq Q_1 \leq Q$ ,*

$$\sum_{q \leq Q} \sum'_{\chi \pmod{q}} F(\chi) \ll \frac{\mathcal{L}Q}{Q_1} \sum_{Q_1 \leq q_1 < 2Q_1} \sum_{\psi \pmod{q_1}}^* F(\psi).$$

*Proof.* We recall that  $\hat{\chi}$  is the primitive character that induces  $\chi$ , so that  $F(\hat{\chi})$  may be quite different from  $F(\chi)$ .

The left-hand side of the claimed inequality is

$$\sum_{q_1 \leq Q} \sum_{\psi \pmod{q_1}}^* F(\psi) \sum_{\substack{\chi \pmod{q} \\ q \leq Q, q_1 | q \\ \psi \text{ induces } \chi}} 1 \leq \sum_{q_1 \leq Q} \sum_{\psi \pmod{q_1}}^* F(\psi) \frac{Q}{q_1}.$$

The lemma follows on applying a splitting-up argument to  $q_1$ .  $\square$

**Lemma 9.** Let  $f(j)$  ( $j \geq 1$ ) be a periodic function with period  $q$ ,

$$S(f, n) = \sum_{j=1}^n f(j) e\left(-\frac{nj}{q}\right),$$

$F > 0$ , and  $R \geq 1$ . Let  $H(y)$  be a real function with  $H'(y)$  monotonic and

$$|H'(y)| \leq Fy^{-1}$$

for  $R \leq y \leq 2R$ . Then for  $J = [R, R']$  with  $R < R' \leq 2R$ ,

$$\sum_{m \in J} f(m)H(m) - q^{-1} \sum_{1 \leq |n| \leq 2FqR^{-1}} S(f, n) \int_J e\left(\frac{ny}{q} + H(y)\right) dy \ll \frac{R|S(f, 0)|}{qF} + \sum_{|n| \in J'} \frac{|S(f, n)|}{n},$$

where

$$J' = [\min\{2FqR^{-1}, q/2\}, \max\{2FqR^{-1}, q\} + q].$$

*Proof.* This is [2, Theorem 8]. □

For a finite sequence  $\{a_k : K \leq k < K'\}$ , set

$$\|a\|_2 = \left( \sum_{K \leq k < K'} |a_k|^2 \right)^{1/2}.$$

**Lemma 10.** Let  $R \geq 1$ ,  $M \geq 1$ ,  $H \geq 1$ . Let  $\beta$  be real and

$$(4.1) \quad \left| \beta - \frac{u_1}{r_1} \right| \leq \frac{H}{r_1^2}$$

where  $r_1 \geq H$  and  $(u_1, r_1) = 1$ . Then for  $M_1 \in \mathbb{N}$ ,

$$(4.2) \quad \sum_{m=M_1+1}^{M_1+M} \min\left(R, \frac{1}{\|m\beta\|}\right) \ll \left(\frac{HM}{r_1} + 1\right) (R + r_1 \log r_1).$$

If  $M < r_1$  and

$$M \left| \beta - \frac{u_1}{r_1} \right| \leq \frac{1}{2r_1},$$

then

$$(4.3) \quad \sum_{m=1}^M \frac{1}{\|m\beta\|} \ll r_1 \log 2r_1.$$

*Proof.* For (4.2), it suffices to show that a block of  $[r_1/H]$  consecutive  $m$ 's contribute

$$\ll R + \sum_{l=1}^{r_1} \frac{r_1}{l}.$$

Writing  $m = m_0 + j$ ,  $1 \leq j \leq [r_1/H]$ ,

$$\left| (m_0 + j)\beta - m_0\beta - \frac{ju_1}{r_1} \right| \leq \frac{jH}{r_1^2} \leq \frac{1}{r_1},$$

so there are  $O(1)$  values of  $j$  for which the bound

$$\|(m_0 + j)\beta\| \geq \frac{1}{2} \left\| m_0\beta + \frac{ju_1}{r_1} \right\|$$

fails. Our block estimate follows immediately.

The argument for (4.3) is similar. In this case,

$$\left| m\beta - \frac{mu_1}{r_1} \right| \leq \frac{1}{2r_1},$$

if  $1 \leq m \leq M$ . Therefore, the left-hand side of (4.3) can be estimated by  $\sum_{l=1}^{r_1} r_1/l$ .  $\square$

**Lemma 11.** *Let  $N < N' \leq 2N$ ,  $MK \asymp N$ ,  $N \geq K \geq M \geq 1$ . Suppose that*

$$(4.4) \quad \left| \gamma - \frac{u}{r} \right| \leq \frac{H}{r^2}, \quad (u, r) = 1, \quad H \leq r \leq N.$$

*Let  $(a_m)_{M \leq m < 2M}$ ,  $(b_k)_{K \leq k < 2K}$  be two sequences of complex numbers. Then*

$$(4.5) \quad S := \sum_{Q \leq q < 2Q} \sum_{\chi \bmod q} \left| \sum_{\substack{M \leq m < 2M \\ N \leq mk < N'}} \sum_{K \leq k < 2K} a_m b_k \chi(mk) e(\gamma mk) \right|$$

*satisfies the bound*

$$S \ll \|a\|_2 \|b\|_2 \mathcal{L}^{3/2} D^{1/2} \left( Q^2 M^{1/2} + \frac{Q^{3/2} H^{1/2} N^{1/2}}{r^{1/2}} + Q^{3/2} H^{1/2} K^{1/2} + Q^{3/2} r^{1/2} \right),$$

*where*

$$D = \max_{n < N} \#\{q \in [Q, 2Q) : n = lq\}.$$

*Proof.* Let  $S'$  be the sum obtained from  $S$  by removing the condition  $N \leq mk < N'$ . It suffices to prove the same bound, with  $\mathcal{L}^{1/2}$  in place of  $\mathcal{L}^{3/2}$ , for  $S'$ , since the condition can be restored at the cost of a factor of  $\mathcal{L}$ . See [8, Section 3.2].

We have

$$S' \leq \sum_{Q \leq q < 2Q} \sum_{\chi \bmod q} \sum_{M \leq m < 2M} |a_m| \left| \sum_{K \leq k < 2K} b_k \chi(k) e(\gamma mk) \right| = \sum_q S_q,$$

say. We may also assume that  $b_k = 0$  if  $(k, q) > 1$ . By Cauchy's inequality, and with summations subject to the obvious restrictions on  $m$ ,  $k_1$  and  $k_2$ ,

$$S_q^2 \leq \varphi(q) \|a\|_2^2 \sum_{\chi \bmod q} \sum_m \sum_{k_1} \sum_{k_2} b_{k_1} \bar{b}_{k_2} \chi(k_1) \bar{\chi}(k_2) e(\gamma m(k_1 - k_2)).$$

Bringing the sum over  $\chi$  inside we see that the right-hand side of the above is

$$\varphi(q)^2 \|a\|_2^2 \sum_{\substack{k_1, k_2 \\ k_1 \equiv k_2 \bmod q}} b_{k_1} \bar{b}_{k_2} \sum_m e(\gamma m(k_1 - k_2)) \leq \varphi(q)^2 \|a\|_2^2 \sum_{k_1} |b_{k_1}|^2 \sum_{k_1 \equiv k_2 \bmod q} \left| \sum_m e(\gamma m(k_1 - k_2)) \right|$$

upon using the parallelogram rule

$$|b_{k_1} b_{k_2}| \leq \frac{1}{2} (|b_{k_1}|^2 + |b_{k_2}|^2).$$

Now summing the geometric sum over  $m$  and then summing over  $q$ , we see that

$$(4.6) \quad \sum_{Q \leq q < 2Q} S_q^2 \ll Q^3 \|a\|_2^2 \|b\|_2^2 M + Q^2 \|a\|_2^2 \|b\|_2^2 \sum_{Q \leq q < 2Q} \sum_{1 \leq l < K/q} \min \left( M, \frac{1}{\|\gamma l q\|} \right).$$

Now we combine the variables  $l$  and  $q$  and then apply (4.2), leading to

$$\begin{aligned} \sum_{Q \leq q < 2Q} S_q^2 &\ll Q^3 \|a\|_2^2 \|b\|_2^2 M + Q^2 \|a\|_2^2 \|b\|_2^2 D \left( \frac{HK}{r} + 1 \right) (M + r \log r) \\ &\ll \|a\|_2^2 \|b\|_2^2 \left( Q^3 M + \mathcal{L} Q^2 D \left( \frac{HN}{r} + HK + M + r \right) \right). \end{aligned}$$

The desired bound for  $S'$  follows by another application of Cauchy's inequality.  $\square$

**Lemma 12.** *Under the hypotheses of Lemma 11, suppose that  $4MQ < N$ ,  $b_k = 1$  for  $K \leq k < 2K$  and  $|a_m| \leq 1$  for  $M \leq m < 2M$ . Define  $D$  as in Lemma 11. Then*



(i) We have

$$S \ll Q^{3/2} \mathcal{L} D \left( \frac{QMH}{r} + 1 \right) \left( \frac{K}{Q} + r \right).$$

(ii) If  $4MQ < r$  and

$$4MQ \left| \gamma - \frac{u}{r} \right| \leq \frac{1}{2r},$$

then

$$S \ll \mathcal{L} D Q^{3/2} r.$$

*Proof.* Let  $I_m$  (here and after) denote a subinterval of  $[N/m, N'/m)$ . We have

$$S \leq QS^* + S^{**},$$

where, for a suitably chosen nonprincipal  $\chi_q \pmod{q}$ ,

$$S^* = \sum_{Q \leq q < 2Q} \sum_{M \leq m < 2M} \left| \sum_{k \in I_m} \chi_q(k) e(\gamma mk) \right|$$

and

$$S^{**} = \sum_{Q \leq q < 2Q} \sum_{M \leq m < 2M} \left| \sum_{k \in I_m} \chi_0(k) e(\gamma mk) \right|.$$

To prove part (i), it suffices to show that

$$S^* \ll Q^{1/2} \mathcal{L} D \left( \frac{QMH}{r} + 1 \right) \left( \frac{K}{Q} + r \right)$$

and

$$S^{**} \ll Q \mathcal{L} D \left( \frac{QMH}{r} + 1 \right) \left( \frac{K}{Q} + 1 \right).$$

We give the proof for  $S^*$ ; the proof for  $S^{**}$  is similar.

Given  $q$  and  $m$ , Lemma 9 together with, using the notation from Lemma 9,

$$|S(\chi, q)| \leq \sqrt{q}$$

(see Chapter 9 of [6]) gives

$$\begin{aligned} \sum_{k \in I_m} \chi_q(k) e(\gamma mk) - \frac{1}{q} \sum_{1 \leq |n| \ll Mq} S(\chi_q, n) \int_{I_m} e \left( \left( \frac{n}{q} + \gamma m \right) y \right) dy \\ \ll q^{-1/2} M^{-1} + q^{1/2} \sum_{1 \leq n \ll Mq} n^{-1} \ll q^{1/2} \mathcal{L}. \end{aligned}$$

Therefore

$$\sum_{k \in I_m} \chi_q(k) e(\gamma mk) \ll q^{1/2} \mathcal{L} + q^{-1/2} \sum_{1 \leq |n| \ll Mq} \min \left( K, \frac{1}{\left| \gamma m - \frac{n}{q} \right|} \right).$$

Summing over  $m$  and  $q$ ,

$$S^* \ll MQ^{3/2} \mathcal{L} + Q^{1/2} \sum_{Q \leq q < 2Q} \sum_{M \leq m < 2M} \sum_{1 \leq |n| \ll Mq} \min \left( \frac{K}{Q}, \frac{1}{|\gamma mq - n|} \right).$$

The contribution to the right-hand side of the above from  $n$ 's with  $|n - \gamma mq| > 1/2$  is

$$\ll MQ^{3/2} \mathcal{L}.$$

Now combining the variables  $m, q$ ,

$$(4.7) \quad S^* \ll MQ^{3/2} \mathcal{L} + Q^{1/2} D \sum_{MQ \leq m' < 4MQ} \min \left( \frac{K}{Q}, \frac{1}{\|\gamma m'\|} \right).$$

We can now deduce the desired bound for  $S^*$  by applying (4.2).

Now for part (ii), we note that (4.3) is applicable to the reciprocal sum in (4.7) with  $4MQ$  and  $\gamma$  in place of  $M$  and  $\beta$ . Hence

$$S^* \ll MQ^{3/2}\mathcal{L} + Q^{1/2}Dr \log 2r \ll D\mathcal{L}Q^{1/2}r$$

since  $4MQ < r$ . Similarly  $S^{**} \ll D\mathcal{L}Qr$ , and part (ii) follows.  $\square$

**Lemma 13.** *Suppose that*

$$\left| \gamma - \frac{u}{r} \right| \leq \frac{\mathcal{L}^{A+1}}{r^2}$$

with  $(u, r) = 1$  and that  $r^2 \leq N \leq r^2\mathcal{L}^{2A+2}$ . Then

(i) *For  $Q < N^{2/7-\varepsilon}$ ,  $N^{4/7} \ll K \ll N^{5/7}$  and any  $a_m, b_k$  with  $|a_m| \leq \tau(m)^B$ ,  $|b_k| \leq \tau(k)^B$ , where  $B$  is an absolute constant, the sum  $S$  in (4.5) satisfies the bound*

$$(4.8) \quad S \ll QN^{1-\varepsilon/4}.$$

(ii) *For  $Q \leq N^{2/7-\varepsilon}$ ,  $M \ll N^{4/7}$  and  $b_k = 1$  for  $K \leq k < 2K$ ,  $|a_m| \leq 1$  for  $M \leq m < 2M$ , the sum  $S$  in (4.5) satisfies (4.8).*

*Proof.* In order to prove (i), we use Lemma 11. As  $D \ll N^{\varepsilon/15}$ ,

$$SQ^{-1}N^{-1+\varepsilon/4} \ll Q^{-1}N^{-1/2+\varepsilon/3} \left( Q^2N^{3/14} + Q^{3/2}N^{5/14} \right) \ll N^{-1/2+\varepsilon/2} \left( QN^{3/14} + Q^{1/2}N^{5/14} \right) \ll 1.$$

To prove (ii), we break the situation into two cases. If  $K < N^{1-\varepsilon}$ , then by (i) of Lemma 12

$$SQ^{-1}N^{-1+\varepsilon/4} \ll Q^{1/2}N^{-1+\varepsilon/2} \left( N^{1/2} + MQ + \frac{N^{1-\varepsilon}}{Q} \right) \ll N^{1/7-1/2+\varepsilon} + N^{3/7+4/7-1-\varepsilon} + N^{-\varepsilon/2} \ll 1.$$

If  $K \geq N^{1-\varepsilon}$ , then  $M \ll N^\varepsilon$  and (ii) of Lemma 12 is applicable since

$$4MQ \left| \gamma - \frac{u}{r} \right| \ll N^{-1+2/7+\varepsilon}.$$

Hence

$$SQ^{-1}N^{-1+\varepsilon/4} \ll Q^{1/2}N^{-1/2+\varepsilon} \ll 1,$$

giving the desired majorant.  $\square$

**Lemma 14.** *Let  $f$  be an arbitrary complex function on  $[N, 2N)$ . Let  $N < N' \leq 2N$ . The sum*

$$S = \sum_{N \leq n < N'} \Lambda(n)f(n)$$

*can be decomposed into  $O(\mathcal{L}^2)$  sums of the form*

$$\sum_{M < m \leq 2M} a_m \sum_{\substack{K \leq k < 2K \\ N \leq mk < N'}} f(mk) \quad \text{or} \quad \int_N^{N'} \sum_{M \leq m < 2M} a_m \sum_{\substack{k \geq w \\ K \leq k < 2K \\ N \leq mk < N'}} f(mk) \frac{dw}{w}$$

with  $M \leq N^{1/4}$  and  $|a_m| \leq 1$ , together with  $O(\mathcal{L})$  sums of the form

$$\sum_{M < m \leq 2M} a_m \sum_{\substack{K \leq k < 2K \\ N \leq mk < N'}} b_k f(mk)$$

with  $N^{1/2} \leq K \ll N^{3/4}$  and  $\|a\|_2 \|b\|_2 \ll N^{1/2}\mathcal{L}^2$ .

*Proof.* This follows from the arguments in [6, Chapter 24] by taking  $U = V = N^{1/4}$ .  $\square$

We record a special case of [3, Lemma 14]. For more background on the ‘‘Harman sieve’’, see [8].

**Lemma 15.** *Let  $W(n)$  be a complex function with support in  $(N, 2N] \cap \mathbb{Z}$ ,  $|W(n)| \leq N^{1/\varepsilon}$ . For  $r \in \mathbb{N}$ ,  $z \geq 2$ , let*

$$(4.9) \quad S^*(r, z) = \sum_{(n, P(z))=1} W(rn).$$

*Suppose that for some constant  $c > 0$ ,  $0 \leq d \leq 1/2$ , and for some  $Y > 0$ , we have, for any coefficients  $a_m$ ,  $b_k$  with  $|a_m| \leq 1$ ,  $|b_k| \leq \tau(k)$ ,*

$$(4.10) \quad \sum_{m \leq 2N^c} a_m \sum_k W(mk) \ll Y,$$

*and*

$$(4.11) \quad \sum_{N^c \leq m \leq 2N^{c+d}} a_m \sum_k b_k W(mk) \ll Y.$$

*Let  $u_r$  ( $r \leq N^c$ ) be complex numbers with  $|u_r| \leq 1$  and  $u_r = 0$  for  $(r, P(N^\varepsilon)) > 1$ . Then*

$$\sum_{r \leq (2N)^c} u_r S^*(r, (2N)^d) \ll Y \mathcal{L}^3.$$

The following application of Lemma 15 will be used in the proof of Theorem 3. We take

$$(4.12) \quad W(n) = \sum_{Q \leq q < 2Q} \sum_{\chi \bmod q} \eta_\chi \chi(n) e(\gamma n)$$

for  $N \leq n < N'$ ; otherwise,  $W(n) = 0$ . Here  $\eta_\chi$  is arbitrary with  $|\eta_\chi| \leq 1$ .

**Lemma 16.** *Suppose that*

$$\left| \gamma - \frac{u}{r} \right| \leq \frac{\mathcal{L}^{A+1}}{r^2}, \quad (u, r) = 1, \quad N = r^2, \quad 1 \leq Q \leq N^{2/7-\varepsilon}.$$

*Define  $S^*(r, z)$  as above with  $W$  defined in (4.12). Then*

$$\sum_{r \leq (2N)^{4/7}} u_r S^*(r, (2N)^{1/7}) \ll N \mathcal{L}^{-A}$$

*for every  $A > 0$ , provided that  $|u_r| \leq 1$ ,  $u_r = 0$  for  $(r, P(N^\varepsilon)) > 1$ .*

*Proof.* We need to verify (4.10) and (4.11) with  $c = 4/7$ ,  $d = 1/7$  and  $Y = N \mathcal{L}^{-A-3}$ . This is an application of Lemma 13.  $\square$

We now introduce some subsets of  $\mathbb{R}^j$  needed in the proof of Theorem 3. Write  $E_j$  for the set of  $j$ -tuples  $\alpha_j = (\alpha_1, \dots, \alpha_j)$  satisfying

$$\frac{1}{7} \leq \alpha_j < \alpha_{j-1} < \dots < \alpha_1 \leq \frac{1}{2} \quad \text{and} \quad \alpha_1 + \alpha_2 + \dots + \alpha_{j-1} + 2\alpha_j \leq 1.$$

A tuple  $\alpha_j$  is said to be *good* if some subsum of  $\alpha_1 + \dots + \alpha_j$  is in  $[2/7, 3/7] \cup [4/7, 5/7]$  and *bad* otherwise.

We use the notation  $p_j = (2N)^{\alpha_j}$ . For instance, the sum

$$\sum_{\substack{p_1 p_2 n_3 = k \\ (2N)^{1/7} \leq p_2 < p_1 < (2N)^{1/2}}} \psi(n_3, p_2)$$

will be written as

$$\sum_{\substack{p_1 p_2 n_3 = k \\ \alpha_2 \in E_2}} \psi(n_3, p_2).$$

**Lemma 17.** Let  $\gamma, u/r, N, Q$  be as in Lemma 16 and  $E$  be a subset of  $E_j$  defined by a bounded number of inequalities of the form

$$(4.13) \quad c_1\alpha_1 + \cdots + c_j\alpha_j < c_{j+1} \text{ (or } \leq c_{j+1}).$$

Suppose that all points in  $E$  are good and that throughout  $E$ ,  $z_j$  is either the function  $z_j = (2N)^{\alpha_j}$  or the constant  $z_j = (2N)^{1/7}$ . Then for arbitrary  $\eta_\chi$  with  $|\eta_\chi| \leq 1$ ,

$$\sum_{Q \leq q < 2Q} \sum_{\chi \bmod q} \eta_\chi \sum_{\substack{N \leq p_1 \cdots p_j n_{j+1} < N' \\ \alpha_j \in E}} \chi(p_1 \cdots p_j n_{j+1}) e(\gamma p_1 \cdots p_j n_{j+1}) \psi(n_{j+1}, z_j) \ll N \mathcal{L}^{-A},$$

for every  $A > 0$ .

*Proof.* This is a consequence of (i) of Lemma 13. On grouping a subset of the variables as a product  $m = \prod_{i \in S} p_i$ , with  $S \subset \{1, \dots, j\}$ , we obtain a sum  $S$  of the form appearing in (i) of Lemma 13, except that a bounded number of inequalities of the form (4.13) are present. These inequalities may be removed at the cost of a log power, by the mechanism noted earlier. See page 184 of [3] for a few more details of a similar argument. The lemma follows at once.  $\square$

**Lemma 18.** Let  $D = \{(\alpha_1, \alpha_2) \in E_2 : (\alpha_1, \alpha_2) \text{ is bad, } \alpha_1 + 2\alpha_2 > 5/7\}$ . Then

$$X(\mathbb{P}; n) - \sum_{\substack{p_1 p_2 n_3 = n \\ \alpha_2 \in D}} \psi(n_3, p_2) = \varrho_1(n) + \varrho_2(n) + \varrho_3(n) - \varrho_4(n) - \varrho_5(n).$$

Here

$$\begin{aligned} \varrho_1(n) &= \psi(n, (2N)^{1/7}), \quad \varrho_4(n) = \sum_{\substack{p_1 n_2 = n \\ \alpha_1 \in E_1}} \psi(n_2, (2N)^{1/7}), \quad \varrho_2(n) = \sum_{\substack{p_1 p_2 n_3 = n \\ \alpha_2 \in E_2 \setminus D}} \psi(n_3, (2N)^{1/7}), \\ \varrho_5(n) &= \sum_{\substack{p_1 p_2 p_3 n_4 = n \\ \alpha_3 \in E_3 \\ (\alpha_1, \alpha_2) \in E_2 \setminus D}} \psi(n_4, (2N)^{1/7}) \quad \text{and} \quad \varrho_3(n) = \sum_{\substack{p_1 p_2 p_3 p_4 n_5 = n \\ \alpha_4 \in E_4 \\ (\alpha_1, \alpha_2) \in E_2 \setminus D}} \psi(n_5, p_4). \end{aligned}$$

*Proof.* We repeatedly use Buchstab's identity in the form

$$\psi(m, z) = \psi(m, w) - \sum_{\substack{ph=m \\ w \leq p < z}} \psi(h, p) \quad (2 \leq w < z).$$

Thus

$$\begin{aligned} (4.14) \quad X(\mathbb{P}; n) &= \psi(n, (2N)^{1/2}) = \psi(n, (2N)^{1/7}) - \sum_{\substack{(2N)^{1/7} \leq p_1 < (2N)^{1/2} \\ p_1 n_2 = n}} \psi(n_2, p_1) \\ &= \varrho_1(n) - \varrho_4(n) + \sum_{\substack{p_1 p_2 n_3 = n \\ \alpha_2 \in E_2}} \psi(n_3, p_2), \end{aligned}$$

$$X(\mathbb{P}; n) - \sum_{\substack{p_1 p_2 n_3 = n \\ \alpha_2 \in D}} \psi(n_3, p_2) = \varrho_1(n) - \varrho_4(n) + \sum_{\substack{p_1 p_2 n_3 = n \\ \alpha_2 \in E_2 \setminus D}} \psi(n_3, p_2).$$

Continuing the decomposition of the last sum,

$$(4.15) \quad \sum_{\substack{p_1 p_2 n_3 = n \\ \alpha_2 \in E_2 \setminus D}} \psi(n_3, p_2) = \sum_{\substack{p_1 p_2 n_3 = n \\ \alpha_2 \in E_2 \setminus D}} \psi(n_3, (2N)^{1/7}) - \sum_{\substack{p_1 p_2 p_3 n_4 = n \\ \alpha_3 \in E_3 \\ (\alpha_1, \alpha_2) \in E_2 \setminus D}} \psi(n_4, (2N)^{1/7}) + \sum_{\substack{p_1 p_2 p_3 p_4 n_5 = n \\ \alpha_4 \in E_4 \\ (\alpha_1, \alpha_2) \in E_2 \setminus D}} \psi(n_5, p_4).$$

Combining (4.14) and (4.15), we complete the proof of the lemma.  $\square$

**Lemma 19.** *Let  $r, u/r, N$  and  $Q$  be as in Lemma 16 with  $\varrho_1, \dots, \varrho_5$  as in Lemma 18; we have*

$$\sum_{Q \leq q < 2Q} \sum_{\chi \bmod q} \eta_\chi \sum_{N \leq n < N'} \varrho_j(n) \chi(n) e(\gamma n) \ll Q N \mathcal{L}^{-A}$$

for arbitrary  $\eta_\chi$  with  $|\eta_\chi| \leq 1$  and any  $A > 0$ .

*Proof.* This follows from Lemmas 16 and 17 for  $j = 1, 2, 4, 5$  on noting that  $\alpha_1 + \alpha_2 + \alpha_3 \leq \alpha_1 + 2\alpha_2 \leq 5/7$  for  $j = 5$ , so that either  $\alpha_3$  is good or  $\alpha_1 + \alpha_2 + \alpha_3 < 4/7$  (similarly for  $j = 2$ ). For  $j = 3$ , we need to show that each  $\alpha_4$  counted is good. Suppose that some  $\alpha_4$  is bad. We have  $\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 \leq 1$ . Hence  $\alpha_1 + \alpha_2 + \alpha_3 \leq 5/7$  from which we infer that  $\alpha_1 + \alpha_2 + \alpha_3 < 4/7$ . Therefore,  $\alpha_1 + \alpha_2 < 3/7$ . But we know that  $\alpha_1 + \alpha_2 > 2/7$ . This makes  $\alpha_4$  good, a contradiction.  $\square$

## 5. PROOF OF THEOREMS 4 AND 5

*Proof of Theorem 4.* With a suitable choice of  $a_q$ ,  $(a_q, q) = 1$ , we have

$$\begin{aligned} \max_{(a,q)=1} E(N, N', \gamma, q, a) &\leq \sup_I \left| \sum_{\substack{N \leq n < N' \\ \gamma n \in I \bmod 1 \\ n \equiv a_q \bmod q}} \Lambda(n) - |I| \sum_{\substack{N \leq n < N' \\ n \equiv a_q \bmod q}} \Lambda(n) \right| + \left| \sum_{\substack{N \leq n < N' \\ n \equiv a_q \bmod q}} \Lambda(n) - \frac{N' - N}{\varphi(q)} \right| \\ &= T_1(q) + T_2(q), \end{aligned}$$

say. In view of the Bombieri-Vinogradov theorem, we need only bound  $\sum_q T_1(q)$ , which is, applying Lemma 7,

$$\ll \sum_{q \leq N^{1/4-\varepsilon}} \mathcal{L}^{-A-1} \sum_{\substack{N \leq n < N' \\ n \equiv a_q \bmod q}} \Lambda(n) + \sum_{q \leq \min(r, N^{1/4}) N^{-\varepsilon}} \sum_{h \leq \mathcal{L}^{A+1}} \frac{1}{h} \left| \sum_{\substack{N \leq n < N' \\ n \equiv a_q \bmod q}} \Lambda(n) e(\gamma n h) \right|.$$

Let  $H = \mathcal{L}^{A+1}$ . Mindful of the Brun-Titchmarsh inequality, it remains to show that for  $1 \leq h \leq H$ ,

$$\sum_{q \leq \min(N^{1/4}, r) N^{-\varepsilon}} \left| \sum_{\substack{N \leq n < N' \\ n \equiv a_q \bmod q}} \Lambda(n) e(\gamma n h) \right| \ll N \mathcal{L}^{-A-1}.$$

Reducing  $hu/r$  into lowest terms, we need only show that

$$\sum_{q \leq \min(N^{1/4}, r) N^{-\varepsilon/2}} \eta_q \sum_{\substack{N \leq n < N' \\ n \equiv a_q \bmod q}} \Lambda(n) e(\gamma n) \ll N \mathcal{L}^{-A-1}$$

under the modified hypothesis (4.4) on  $\gamma$  (with  $H = \mathcal{L}^{A+1}$ ), whenever  $|\eta_q| \leq 1$ .

Using Lemma 14, it suffices to show that

$$(5.1) \quad \sum_{q \leq \min(N^{1/4}, r) N^{-\varepsilon/2}} \eta_q \sum_{\substack{M \leq m < 2M \\ N \leq mk < N' \\ mk \equiv a_q \bmod q}} \sum_{K \leq k < 2K} a_m b_k e(\gamma mk) \ll N \mathcal{L}^{-A-3}$$

under either of the following sets of conditions.

- (a)  $\|a\|_2 \|b\|_2 \ll N^{1/2} \mathcal{L}^2$ ,  $N^{1/2} \leq K \leq N^{3/4}$ ;
- (b)  $|a_m| \leq 1$ ,  $b_k = 1$  for  $k \in I_m \subset [K, 2K)$ ,  $b_k = 0$  otherwise,  $M \leq N^{1/4}$ .

We use Dirichlet characters to detect the congruence relation in (5.1) and we require the estimate

$$\sum_{q \leq \min(N^{1/4}, r) N^{-\varepsilon/2}} \frac{\eta_q}{\varphi(q)} \sum_{\chi \bmod q} \bar{\chi}(a_q) \sum_{\substack{M \leq m < 2M \\ N \leq mk < N'}} \sum_{K \leq k < 2K} a_m b_k \chi(mk) e(\gamma mk) \ll N \mathcal{L}^{-A-4}.$$

It suffices to show that

$$(5.2) \quad S := \sum_{Q \leq q < 2Q} \sum_{\chi \bmod q} \left| \sum_{\substack{M \leq m < 2M \\ N \leq mk < N'}} \sum_{K \leq k < 2K} a_m b_k \chi(mk) e(\gamma mk) \right| \ll QN\mathcal{L}^{-A-6}$$

for  $Q \leq \min(N^{1/4}, r)N^{-\varepsilon/2}$ .

In case (a), we apply Lemma 11, which gives

$$\begin{aligned} S &\ll N^{1/2+\varepsilon/6} \left( Q^2 M^{1/2} + \frac{Q^{3/2} N^{1/2}}{r^{1/2}} + Q^{3/2} K^{1/2} + Q^{3/2} r^{1/2} \right) \\ &\ll N^{3/4+\varepsilon/6} Q^2 + \frac{N^{1+\varepsilon/6} Q^{3/2}}{r^{1/2}} + Q^{3/2} N^{7/8+\varepsilon/6}. \end{aligned}$$

Each one of these three terms is  $\ll QN\mathcal{L}^{-A-6}$  as

$$\begin{aligned} N^{3/4+\varepsilon/6} Q^2 (QN\mathcal{L}^{-A-6})^{-1} &\ll QN^{-1/4+\varepsilon/5} \ll 1, \\ N^{1+\varepsilon/6} Q^{3/2} r^{-1/2} (QN\mathcal{L}^{-A-6})^{-1} &\ll Q^{1/2} N^{\varepsilon/4} r^{-1/2} \ll 1, \end{aligned}$$

since  $Q \leq rN^{-\varepsilon/2}$ , and

$$N^{7/8+\varepsilon/6} Q^{3/2} (QN\mathcal{L}^{-A-6})^{-1} \ll N^{-1/8+\varepsilon/5} Q^{1/2} \ll 1.$$

In case (b), we use Lemma 12. Suppose that  $K < N^{1-\varepsilon/4}$ ; (i) of Lemma 12 gives

$$S \ll Q^{3/2} N^{\varepsilon/6} \left( \frac{N}{r} + QM + \frac{K}{Q} + r \right).$$

Each of the above four terms is  $\ll QN\mathcal{L}^{-A-6}$ , since

$$\begin{aligned} \frac{Q^{3/2} N^{1+\varepsilon/6}}{r} (QN\mathcal{L}^{-A-6})^{-1} &\ll Q^{1/2} r^{-1} N^{\varepsilon/5} \ll 1, \\ Q^{5/2} N^{\varepsilon/6} M (QN\mathcal{L}^{-A-6})^{-1} &\ll Q^{3/2} N^{-3/4+\varepsilon/5} \ll 1, \\ Q^{1/2} N^{\varepsilon/6} K (QN\mathcal{L}^{-A-6})^{-1} &\ll KN^{-1+\varepsilon/4} \ll 1 \end{aligned}$$

and

$$Q^{3/2} N^{\varepsilon/6} r (QN\mathcal{L}^{-A-6})^{-1} \ll Q^{1/2} N^{-1/4+\varepsilon/5} \ll 1.$$

Now suppose that  $K \geq N^{1-\varepsilon/4}$ . Then

$$4MQ \ll QN^{\varepsilon/4}, \text{ thus } 4MQ < r$$

and

$$4MQr \left| \gamma - \frac{u}{r} \right| \ll MQN^{-3/4}, \text{ hence } 4MQr \left| \gamma - \frac{u}{r} \right| \leq \frac{1}{2}.$$

So (ii) of Lemma 12 gives comfortably:

$$S \ll N^{\varepsilon} Q^{3/2} r \ll QN\mathcal{L}^{-A-6},$$

completing the proof. □

*Proof of Theorem 5.* We first show that the contribution to the sum in (1.10) from  $q \leq \mathcal{L}^{A+1}$  is

$$\ll N^2 \mathcal{L}^{-A} \ll NR.$$

Since, for some  $Q \leq \mathcal{L}^{A+1}$ ,

$$\sum_{q \leq \mathcal{L}^{A+1}} \sum_{\substack{a=1 \\ (a,q)=1}}^q E^2 \ll N \sum_{q \leq \mathcal{L}^{A+1}} \frac{1}{\varphi(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^q E(N, N', \gamma, q, a) \ll \frac{N\mathcal{L}}{Q} \sum_{Q \leq q < 2Q} \max_{(a,q)=1} E(N, N', \gamma, q, a),$$

it suffices to show for this  $Q$  that

$$(5.3) \quad \sum_{Q \leq q < 2Q} \max_{(a,q)=1} E(N, N', \gamma, q, a) \ll QN\mathcal{L}^{-A-1}.$$

We may suppose that  $A$  is large. Arguing as in the proof of Theorem 4, we need only show that (5.2) follows from either (a) or (b). By Dirichlet's theorem, there is a rational approximation  $b/r$  to  $\gamma$  satisfying (1.7). For any  $\eta > 0$ ,

$$N^{-3/4} \geq \|\gamma r\| \gg \exp(-r^\eta),$$

hence  $r \gg \mathcal{L}^{5A}$ . Now we apply Lemma 11 to prove the desired bound under (a). Since  $D \leq Q \leq \mathcal{L}^{A+1}$ , the term

$$\|a\|_2 \|b\|_2 \mathcal{L}^2 D^{1/2} Q^{3/2} H^{1/2} N^{1/2} r^{-1/2}$$

presents no difficulty; the other terms are clearly all small enough. For the bound under (b), a similar remark applies to Lemma 12 and the terms

$$Q^{3/2} \mathcal{L} D N H r^{-1}$$

if  $K < N^{1-\varepsilon/4}$  and

$$\mathcal{L} D Q^{3/2} r$$

if  $K \geq N^{1-\varepsilon/4}$ . This establishes (5.3).

It remains to examine the contribution to the sum in (1.10) from  $q \in [Q, 2Q]$  with  $\mathcal{L}^{A+1} \leq Q \leq R$ . We have

$$\begin{aligned} \sum_{Q \leq q < 2Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q E(N, N', \gamma, q, a)^2 &\ll \sum_q \sum_a \sup_I \left| \sum_{\substack{N < n \leq N' \\ \{\gamma n\} \in I \\ n \equiv a \pmod q}} \Lambda(n) - |I| \sum_{\substack{N < n \leq N' \\ n \equiv a \pmod q}} \Lambda(n) \right|^2 \\ &\quad + \sum_q \sum_a \left( \sum_{\substack{N < n \leq N' \\ n \equiv a \pmod q}} \Lambda(n) - \frac{N' - N}{\varphi(q)} \right)^2 = T_1(Q) + T_2(Q), \end{aligned}$$

say. Since  $T_2(Q)$  is covered by a slight variant of the discussion in [6, Chapter 29], we focus our attention on  $T_1(Q)$ . By Lemma 7,

$$\begin{aligned} T_1(Q) &\ll \sum_{Q \leq q < 2Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q \mathcal{L}^{-2A} \left( \sum_{\substack{N < n \leq N' \\ n \equiv a \pmod q}} \Lambda(n) \right)^2 + \sum_{Q \leq q < 2Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left( \sum_{h \leq \mathcal{L}^A} \frac{1}{h} \left| \sum_{\substack{N < n \leq N' \\ n \equiv a \pmod q}} \Lambda(n) e(\gamma n h) \right| \right)^2 \\ &= T_3(Q) + T_4(Q), \end{aligned}$$

say. The Brun-Titchmarsh Theorem gives a satisfactory bound for  $T_3(Q)$ . Applying Cauchy's inequality to  $T_4(Q)$ , we get

$$\begin{aligned} T_4(Q) &\leq \left( \sum_{h \leq \mathcal{L}^A} \frac{1}{h} \right) \sum_{h \leq \mathcal{L}^A} \frac{1}{h} \sum_{Q \leq q < 2Q} \sum_{\substack{a=1 \\ (a,q)=1}}^q \left| \sum_{\substack{N < n \leq N' \\ n \equiv a \pmod q}} \Lambda(n) e(\gamma n h) \right|^2 \\ &\ll (\log \mathcal{L})^2 \sum_{Q \leq q < 2Q} \frac{1}{\varphi(q)} \sum_{\chi \pmod q} \left| \sum_{N < n \leq N'} \Lambda(n) \chi(n) e(\gamma n h) \right|^2, \end{aligned}$$

for some  $h \leq \mathcal{L}^A$ . From this point, we can conclude the proof by following, with slight changes, the argument in [6, pp. 170-171].  $\square$

## 6. PROOF OF THEOREMS 2 AND 3

*Proof of Theorem 2.* Let  $\gamma = \alpha^{-1}$  and  $N \geq C_1(\alpha, t)$ ,  $0 < \varepsilon < C_2(\alpha, t)$ . By Dirichlet's theorem, there is a reduced fraction  $b/r$  satisfying (1.7). Our hypothesis on  $\alpha$  implies that

$$N^{-3/4} \geq \|\gamma r\| \gg r^{-3}, \quad r \gg N^{1/4}.$$

Let  $h_1'', \dots, h_l''$  be the first  $l$  primes in  $(l, \infty)$ . Any translate

$$\mathcal{H} = \{h_1', \dots, h_k'\} + h, \quad h \in \mathbb{N}$$

with  $\{h_1', \dots, h_k'\} \subset \{h_1'', \dots, h_l''\}$ , is an admissible set. Using (i) of Lemma 7, we choose  $h_1', \dots, h_k'$  so that

$$(6.1) \quad k \geq \varepsilon \gamma l$$

and for some real  $\eta$ ,

$$-\gamma h_m' \in (\eta, \eta + \varepsilon \gamma) \pmod{1}$$

for every  $m = 1, \dots, k$ . Now choose  $h \in \mathbb{N}$ ,  $h \ll_\gamma 1$  so that

$$h\gamma \in (\eta - \varepsilon \gamma, \eta) \pmod{1}.$$

Thus, writing  $h_m = h_m' + h$ , we have

$$-\gamma h_m = -\gamma h_m' - \gamma h \in (0, 2\varepsilon \gamma) \pmod{1}.$$

We apply Theorem 1 to the set

$$\mathcal{A} = \{n \in [N, 2N) : \gamma m \in I \pmod{1}\}$$

where  $I = (\gamma\beta - \gamma, \gamma\beta)$ , taking  $q_0 = q_1 = 1$ ,  $s = 1$ ,  $\varrho(n) = X(\mathbb{P}; n)$ ,  $\theta = 1/4 - \varepsilon$ ,  $b = 1 - 2\varepsilon$ ,

$$Y = \gamma N, \quad Y_{1,m} = l_m \int_N^{2N} \frac{1}{\log t} dt = \frac{l_m Y}{\mathcal{L} \gamma} (1 + o(1)).$$

Here  $J_m, l_m$  are the interval  $J$  and its length  $l$  in Lemma 6 (with  $\varepsilon \gamma$  in place of  $\varepsilon$ ), so that

$$\gamma > l_m > \gamma(1 - 2\varepsilon).$$

Since (1.2) can be proved in a similar (but simpler) fashion to (1.5), we only show that (1.5) holds. We can rewrite this in the form

$$(6.2) \quad \sum_{q \leq x^{1/4-\varepsilon}} \mu^2(q) \tau_{3k}(q) \left| \sum_{\substack{N+h_m \leq p < 2N \\ p \equiv a_q \pmod{q} \\ \gamma p \in J_m \pmod{1}}} 1 - \frac{l_m}{\varphi(q)} \int_N^{2N} \frac{dt}{\log t} \right| \ll N \mathcal{L}^{-k-\varepsilon}.$$

The function  $E(N, N', \gamma, q, a)$  appearing in Theorem 4 is not quite in the form that we need. However, discarding prime powers and using partial summation in the standard way, we readily deduce a variant of (6.2) from Theorem 4, in which  $N \mathcal{L}^{-A}$  appears in place of  $N \mathcal{L}^{-k-\varepsilon}$ , and the weight  $\mu^2(q) \tau_{3k}(q)$  is absent. We then obtain (6.2) by using Cauchy's inequality; see [9, (5.20)] for a very similar computation.

We are now in a position to use Theorem 1, obtaining a set of  $t$  primes in  $\mathcal{A} \cap [N, 2N)$ , which of course have the form  $[a\alpha + \beta]$ , with

$$D(\mathcal{S}) \leq h_k - h_1 \leq h_l''$$

provided that

$$(6.3) \quad M_k > \frac{2t - 2}{(1 - 2\varepsilon)(1/4 - \varepsilon)}.$$

We take  $l$  to be the least integer with

$$\log(\varepsilon \gamma l) \geq \frac{2t - 2}{(1 - 2\varepsilon)(1/4 - \varepsilon)} + C$$



for a suitable absolute constant  $C$ , so that (6.3) follows from (6.1) and (1.11). Therefore,

$$\gamma l \ll \exp(8t), \quad l \ll \alpha \exp(8t), \quad D(\mathcal{S}) \ll l \log l \ll \alpha(t + \log \alpha) \exp(8t),$$

completing the proof.  $\square$

In the proof of Theorem 3, we shall need the following.

**Lemma 20.** *Let  $D$  be as in Lemma 18 and let  $\omega_0(t)$  denote Buchstab's function.*

(i) *The points of  $D$  lie in two triangles  $A_1, A_2$ , where  $A_1$  has vertices*

$$\left(\frac{5}{21}, \frac{5}{21}\right), \left(\frac{2}{7}, \frac{3}{14}\right), \left(\frac{2}{7}, \frac{2}{7}\right)$$

*and  $A_2$  has vertices*

$$\left(\frac{1}{2}, \frac{3}{14}\right), \left(\frac{3}{7}, \frac{2}{7}\right), \left(\frac{1}{2}, \frac{1}{4}\right).$$

(ii) *For  $j = 1, 2$ , let*

$$I_j = \int_{A_j} \frac{1}{\alpha_1 \alpha_2^2} \omega_0\left(\frac{1 - \alpha_1 - \alpha_2}{\alpha_2}\right) d\alpha_1 d\alpha_2.$$

*Then  $I_1 < 0.03925889$  and  $I_2 < 0.0566295$ .*

*Proof.* Let  $(\alpha_1, \alpha_2) \in D$ . If  $\alpha_1 + \alpha_2 > 5/7$ , then we have

$$\alpha_1 + \alpha_2 > \frac{5}{7}, \quad \alpha_1 + 2\alpha_2 \leq 1, \quad \alpha_1 \leq \frac{1}{2}.$$

This defines a triangle which is easily verified to be  $A_2$ . If  $\alpha_1 + \alpha_2 \leq 5/7$ , then as  $\alpha_2$  is bad, we have in turn

$$\alpha_1 + \alpha_2 < \frac{4}{7}, \quad \alpha_1 < \frac{3}{7}, \quad \alpha_1 < \frac{2}{7}.$$

Altogether, we have

$$\alpha_1 + 2\alpha_2 > \frac{5}{7}, \quad \alpha_1 < \frac{2}{7}, \quad \alpha_2 < \alpha_1.$$

This defines a triangle which we can verify to be  $A_1$ . This proves (i). Now (ii) requires a computer calculation, which was kindly carried out by Andreas Weingartner.  $\square$

*Proof of Theorem 3.* With a different value of  $l$ , we choose  $h_1'', \dots, h_l''$  and  $h_1, \dots, h_k$  exactly as in the proof of Theorem 2. In applying Theorem 1, we also take  $I, \mathcal{A}, q_0, q_1, Y, J_m, l_m$  as in that proof, but now  $\theta = 2/7 - \varepsilon$ ,  $s = 5$ ,  $a = 3$ ; the functions  $\varrho_1(n), \dots, \varrho_5(n)$  are given in Lemma 18.

There is little difficulty in verifying (1.2) by a similar but simpler version of the proof of (1.5). So we concentrate on (1.5). We recall that this can be rewritten as

$$(6.4) \quad \sum_{q \leq x^\theta} \mu^2(q) \tau_{3k}(q) \left| \sum_{\substack{n \equiv a_q \pmod{q} \\ \gamma n \in J_m \pmod{1} \\ N + h_m \leq n < 2N}} \varrho_g(n) - \frac{Y_{g,m}}{\varphi(q)} \right| \ll N \mathcal{L}^{-k-\varepsilon}.$$

We define  $Y_{g,m}$  by

$$Y_{g,m} = l_m \sum_{N \leq n < 2N} \varrho_g(n).$$

It is well known that

$$(6.5) \quad Y_{g,m} = \frac{l_m c_g N}{\mathcal{L}} (1 + o(1)),$$

where  $c_g$  is given by a multiple integral. In fact, we have

$$c_1 + c_2 + c_3 - c_4 - c_5 = 1 - \int_{\alpha_2 \in D} \frac{1}{\alpha_1 \alpha_2^2} \omega_0 \left( \frac{1 - \alpha_1 - \alpha_2}{\alpha_2} \right) d\alpha_1 d\alpha_2.$$

Similar calculations are found in [8, Chapter 1].

Fix  $m$  and  $g$ . By analogy with the proof of Theorem 4, we can obtain (1.5) by showing

$$(6.6) \quad \sum_{q \leq N^{2/7-\varepsilon}} \left| \sum_{\substack{N \leq n < N' \\ n \equiv a_q \pmod{q}}} \varrho_g(n) - \frac{1}{\varphi(q)} \sum_{N \leq n < N'} \varrho_g(n) \right| \ll N \mathcal{L}^{-A}$$

for every  $A > 0$  and

$$(6.7) \quad \sum_{q \leq N^{2/7-\varepsilon}} \left| \sum_{\substack{N \leq n < N' \\ n \equiv a_q \pmod{q}}} \varrho_g(n) e(\gamma n h) \right| \ll N \mathcal{L}^{-A}$$

for  $1 \leq h \leq \mathcal{L}^{A+1}$  and for every  $A > 0$ . Again adapting the argument of Theorem 4, we see that (6.7) is a consequence of Lemma 19.

For (6.6), it suffices to show, recalling Lemma 8, that for arbitrary  $\eta_\chi \ll 1$  and  $Q \leq N^{2/7-\varepsilon}$ ,

$$(6.8) \quad \sum_{Q \leq q < 2Q} \sum_{\chi \pmod{q}}^* \eta_\chi \sum_{N \leq n < N'} \varrho_g(n) \chi(n) \ll Q N \mathcal{L}^{-A}$$

for every  $A > 0$ . This can be readily deduced from the Siegel-Walfisz theorem for  $Q \leq \mathcal{L}^{2A}$ , so we assume that  $Q > \mathcal{L}^{2A}$ .

We apply Lemma 15 with

$$W(n) = \sum_{Q \leq q < 2Q} \sum_{\chi \pmod{q}}^* \eta_\chi \chi(n)$$

if  $N \leq n < N'$  and  $W(n) = 0$  otherwise.

For example, when  $g = 3$ , the left-hand side of (6.8) is

$$\sum_{\substack{N \leq p_1 p_2 p_3 n_4 < N' \\ (n_4, P((2N)^{1/7})=1 \\ \alpha_3 \in E_3 \\ (\alpha_1, \alpha_2) \in E_2 \setminus D}} W(p_1 p_2 p_3 n_4) = \sum_{\substack{\alpha_3 \in E_3 \\ (\alpha_1, \alpha_2) \in E_2 \setminus D}} S^*(p_1 p_2 p_3, (2N)^{1/7}).$$

We shall show that (4.10) and (4.11) hold with  $Y = Q N \mathcal{L}^{-A-3}$ ,  $c = 4/7$  and  $d = 1/7$ . (We could reduce the constraints on  $c$  and  $d$ , but that would not be useful in the present context.) Once we have done this, we can follow the proof of Lemma 19 to prove (6.8).

To prove (4.10), we use the Polya-Vinogradov bound for character sums to obtain

$$\begin{aligned} \sum_{m \leq 2N^{4/7}} \sum_k W(nk) &= \sum_{m \leq 2N^{4/7}} a_m \sum_{Q \leq q < 2Q} \sum_{\substack{\chi \pmod{q} \\ N \leq mk < N'}}^* \eta_\chi \chi(mk) \\ &\ll \mathcal{L} \sum_{m \leq 2N^{4/7}} \sum_{Q \leq q < 2Q} q^{1/2} \ll \mathcal{L} Q^{3/2} N^{4/7-\varepsilon} \ll Q N \mathcal{L}^{-A-3}. \end{aligned}$$

Now to prove (4.11), we note that by the method of [8, Section 3.2] mentioned earlier, it suffices to show that

$$\sum_{M \leq m < 2M} a_m \sum_{K \leq k < 2K} b_k W(mk) \ll QN\mathcal{L}^{-A}$$

whenever  $|a_m| \leq 1$  and  $|b_k| \leq \tau(k)$ ,  $N^{4/7} \ll M \ll N^{5/7}$ ,  $MK \asymp N$ . That is, it suffices to show that

$$(6.9) \quad \sum_{Q \leq q < 2Q} \sum_{\chi \bmod q}^* \left| \sum_{M \leq m < 2M} a_m \chi(m) \right| \left| \sum_{K \leq k < 2K} b_k \chi(k) \right| \ll QN\mathcal{L}^{-A}.$$

Following the proof of (6) in [6, Chapter 28], the left-hand side of (6.9) is

$$\ll \mathcal{L}(M + Q^2)^{1/2} (K + Q^2)^{1/2} \|a\|_2 \|b\|_2 \ll \mathcal{L}^3 \left( N^{1/2} + M^{1/2}Q + Q^2 \right) N^{1/2} \ll QN\mathcal{L}^{-A},$$

since  $\mathcal{L}^3 Q^{-1}N \ll \mathcal{L}^{3-A}N$ ,  $\mathcal{L}^3 M^{1/2}N^{1/2} \ll \mathcal{L}^3 N^{6/7} \ll N\mathcal{L}^{-A}$  and  $\mathcal{L}^3 QN^{1/2} \ll \mathcal{L}^3 N^{11/14} \ll N\mathcal{L}^{-A}$ . This proves (1.5) with the present choice of  $\mathcal{A}$ ,  $Y_{g,m}$ , etc.

Applying Theorem 1, we find that there is a set  $\mathcal{S}$  of  $t$  primes in  $\mathcal{A}$  (and thus of the form  $[\alpha m + \beta]$ ) having diameter

$$D(\mathcal{S}) \leq h_k - h_1 \ll l \log l$$

provided that

$$M_k > \frac{2t - 2}{b(2/7 - \varepsilon)}.$$

Here  $b$  must have the property

$$b_{1,m} + b_{2,m} + b_{3,m} - b_{4,m} - b_{5,m} \geq b > 0;$$

that is,

$$l_m(c_1 + c_2 + c_3 - c_4 - c_5) \geq b\gamma > 0.$$

We can choose

$$b = (1 - 2\varepsilon) \left( 1 - \int_{\alpha_2 \in D} \frac{1}{\alpha_1 \alpha_2^2} \omega_0 \left( \frac{1 - \alpha_1 - \alpha_2}{\alpha_2} \right) d\alpha_1 d\alpha_2 \right).$$

Using Lemma 20, we see that

$$b > 0.90411.$$

Now we proceed just as the proof of Theorem 2. We may choose any  $l$  for which

$$\log(\varepsilon\gamma l) \geq \frac{2t - 2}{0.90411(2/7 - \varepsilon)} + C$$

for a suitable constant  $C$ , and now it is a simple matter to deduce that

$$D(\mathcal{S}) < C_4 \alpha (\log \alpha + t) \exp(7.743t),$$

where  $C_4$  is an absolute constant. □

**Acknowledgments.** This work was done while L. Z. held a visiting position at the Department of Mathematics of Brigham Young University (BYU). He wishes to thank the warm hospitality of BYU during his thoroughly enjoyable stay in Provo.

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